

# Singularities of stable varieties

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## 1. INTRODUCTION

The theory of moduli of curves has been extremely successful and part of this success is due to the compactification of the moduli space of smooth projective curves by the moduli space of stable curves. A similar construction is desirable in higher dimensions but unfortunately the methods used for curves do not produce the same results in higher dimensions. In fact, even the definition of what *stable* should mean is not entirely clear a priori. In order to construct modular compactifications of moduli spaces of higher dimensional canonically polarized varieties one must understand the possible degenerations that would produce this desired compactification that itself is a moduli space of an enlarged class of canonically polarized varieties.

The main purpose of the present article is to discuss the relevant issues that arise in higher dimensions and how these lead us to the definition of stable varieties and stable families. Particular emphasis is placed on understanding the singularities of stable varieties including some recent results.

The structure of the article is the following: In §2 and §3 I review the relevant properties of stable curves and their families, including the admissible singularities of the total spaces of stable families. In §4 show how generalizing the properties of the total spaces of stable families leads to the right generalization of stable singularities in higher dimensions. In §5 I review the construction and main properties of canonical sheaves and divisors. §6 is devoted to the singularities of the minimal model program, mainly from a moduli theoretic point of view. In §7 I define some important basic notions and recall some fundamental theorems such as Grothendieck duality and Kodaira vanishing. §8 and §9 are concerned with the definition and basic properties of rational and Du Bois singularities respectively. In §10 I review the most important criteria for rational and Du Bois singularities organized around the principle that a natural morphism in the derived category should admit a left inverse essentially only if it is a quasi-isomorphism combined

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by a push-forward map admitting a section by a trace map. In §11 I review the applications of the results in §10 to stable families and in §12 the state of knowledge about the deformation theory of stable singularities.

Without trying to be comprehensive, here is a list of relevant references on background. In order to study higher dimensional varieties one should be familiar with the main techniques of birational geometry. The standard reference for this is [KM98] and for some more recent results the reader may consult [HK10]. For moduli spaces of higher dimensional smooth varieties a good reference is [Vie95]. For moduli spaces of stable varieties one may refer to [Kol85, KSB88, Kol90]. A light introduction to the ideas involved is contained in [Kov09].

**DEFINITIONS AND NOTATION 1.1.** Let  $k$  be an algebraically closed field of characteristic 0. Unless otherwise stated, all objects will be assumed to be defined over  $k$ . A *scheme* will refer to a scheme of finite type over  $k$  and unless stated otherwise, a *point* refers to a closed point.

For a morphism  $f : Y \rightarrow S$  and another morphism  $T \rightarrow S$ , the symbol  $Y_T$  will denote  $Y \times_S T$ . In particular, for  $t \in S$  I will write  $Y_t = f^{-1}(t)$ . In addition, if  $T = \operatorname{Spec} F$ , then  $Y_T$  will also be denoted by  $Y_F$ .

Let  $X$  be a scheme and  $\mathcal{F}$  an  $\mathcal{O}_X$ -module. The  $m^{\text{th}}$  *reflexive power* of  $\mathcal{F}$  is the double dual (or reflexive hull) of the  $m^{\text{th}}$  tensor power of  $\mathcal{F}$ :

$$\mathcal{F}^{[m]} := (\mathcal{F}^{\otimes m})^{**}.$$

A *line bundle* on  $X$  is an invertible  $\mathcal{O}_X$ -module. A  $\mathbb{Q}$ -*line bundle*  $\mathcal{L}$  on  $X$  is a reflexive  $\mathcal{O}_X$ -module of rank 1 one of whose reflexive power is a line bundle, i.e., there exists an  $m \in \mathbb{N}_+$  such that  $\mathcal{L}^{[m]}$  is a line bundle. The smallest such  $m$  is called the *index* of  $\mathcal{L}$ .

For the advanced reader: whenever I mention Weil divisors, assume that  $X$  is  $S_2$  and think of a *Weil divisorial sheaf*, that is, a rank 1 reflexive  $\mathcal{O}_X$ -module which is locally free in codimension 1. For flatness issues consult [Kol08a, Theorem 2].

For the novice: whenever I mention Weil divisors, assume that  $X$  is normal and adopt the definition [Har77, p.130]. For the adventurous novice: This is mainly interesting for canonical divisors. Read §5.

For a Weil divisor  $D$  on  $X$ , its associated *Weil divisorial sheaf* is the  $\mathcal{O}_X$ -module  $\mathcal{O}_X(D)$  defined on the open set  $U \subseteq X$  by the formula

$$\Gamma(U, \mathcal{O}_X(D)) = \left\{ \frac{a}{b} \mid a, b \in \Gamma(U, \mathcal{O}_X), b \text{ is not a zero divisor} \right. \\ \left. \text{anywhere on } U, \text{ and } D + \operatorname{div}(a) - \operatorname{div}(b) \geq 0 \right\}$$

and made into a sheaf by the natural restriction maps.

A Weil divisor  $D$  on  $X$  is a *Cartier divisor*, if its associated Weil divisorial sheaf,  $\mathcal{O}_X(D)$  is a line bundle. If the associated Weil divisorial sheaf,  $\mathcal{O}_X(D)$  is

a  $\mathbb{Q}$ -line bundle, then  $D$  is a  $\mathbb{Q}$ -Cartier divisor. The latter is equivalent to the property that there exists an  $m \in \mathbb{N}_+$  such that  $mD$  is a Cartier divisor.

The symbol  $\sim$  stands for *linear* and  $\equiv$  for *numerical equivalence* of divisors.

Let  $\mathcal{L}$  be a line bundle on a scheme  $X$ . It is said to be *generated by global sections* if for every point  $x \in X$  there exists a global section  $\sigma_x \in H^0(X, \mathcal{L})$  such that the germ  $\sigma_x$  generates the stalk  $\mathcal{L}_x$  as an  $\mathcal{O}_X$ -module. If  $\mathcal{L}$  is generated by global sections, then the global sections define a morphism

$$\phi_{\mathcal{L}}: X \rightarrow \mathbb{P}^N = \mathbb{P}(H^0(X, \mathcal{L})).$$

$\mathcal{L}$  is called *semi-ample* if  $\mathcal{L}^m$  is generated by global sections for  $m \gg 0$ .  $\mathcal{L}$  is called *ample* if it is semi-ample and  $\phi_{\mathcal{L}^m}$  is an embedding for  $m \gg 0$ . A line bundle  $\mathcal{L}$  on  $X$  is called *big* if the global sections of  $\mathcal{L}^m$  define a rational map  $\phi_{\mathcal{L}^m}: X \dashrightarrow \mathbb{P}^N$  such that  $X$  is birational to  $\phi_{\mathcal{L}^m}(X)$  for  $m \gg 0$ . Note that in this case  $\mathcal{L}^m$  is not necessarily generated by global sections, so  $\phi_{\mathcal{L}^m}$  is not necessarily defined everywhere. I will leave it to the reader to make the obvious adaptation of these notions for the case of  $\mathbb{Q}$ -line bundles.

If it exists, then a *canonical divisor* of a scheme  $X$  is denoted by  $K_X$  and the *canonical sheaf* of  $X$  is denoted by  $\omega_X$ . See §5 for more.

A smooth projective variety  $X$  is of *general type* if  $\omega_X$  is big. It is easy to see that this condition is invariant under birational equivalence between smooth projective varieties. An arbitrary projective variety is of *general type* if so is a desingularization of it.

A projective variety is *canonically polarized* if  $\omega_X$  is ample. Notice that if a smooth projective variety is canonically polarized, then it is of general type.

Further definitions will be given in later sections. In particular, for the definition of *Cohen-Macaulay* and *Gorenstein* see §5.

## 2. STABLE CURVES

First I will recall the definition and main properties of families of stable curves and then subsequently investigate how these may be generalized to higher dimensions.

DEFINITION 2.1. [HM98, 2.12] A *stable curve* is a connected projective curve that

- (2.1.1) has only nodes as singularities; and,
- (2.1.2) has only finitely many automorphisms.

The finiteness condition on the automorphism group is equivalent to either one of the following:

- (2.1.2a) Every smooth rational component of the curve meets the other components in at least 3 points.
- (2.1.2b) The dualizing sheaf of the curve is ample.

With respect to (2.1.2b) note that nodes are local complete intersections and hence a stable curve is Gorenstein by definition. In particular its dualizing sheaf exists and it is a line bundle, and hence it makes sense to ask whether it is ample.

The fact that the moduli functor of stable curves gives a good compactification of the moduli functor of smooth curves hinges on the stable reduction theorem:

**Theorem 2.2.** [HM98, 3.47], [KKMSD73] *Let  $B$  be a smooth curve,  $0 \in B$  a point, and  $B^\circ = B \setminus \{0\}$ . Let  $X^\circ \rightarrow B^\circ$  be a flat family of stable curves of genus  $\geq 2$ . Then there exists a branched cover  $B' \rightarrow B$  totally ramified over  $0$  and a family  $X' \rightarrow B'$  of stable curves extending the fiber product  $X^\circ \times_{B^\circ} B'$ . Moreover, any two such extensions are dominated by a third. In particular, their special fibers, that is, the preimage of  $0$  in  $B'$ , are isomorphic.*

Note that being a family of stable curves implies that  $X' \rightarrow B'$  does not have any multiple fibers. On the other hand, one cannot expect to have a smooth total space,  $X'$ , for this family, although its singularities are the mildest possible: In general  $X'$  will have Du Val singularities (of type  $A$ ). This follows from an explicit computation of the versal deformation space of a node. These singularities may be resolved by successive blowing ups resulting in an exceptional divisor consisting of a chain of rational curves, each appearing with multiplicity 1 in the fiber of the blown up surface over the point  $0 \in B$ . This leads to semi-stable reduction where one only requires the curves in the family to be semi-stable, that is, instead of (2.1.2a) one only requires that every smooth rational component of the curve meets the other components in at least 2 points, but in exchange one obtains that one may require the total space of the family be smooth.

In the next statement I collect the ideas from these observations that will be important in our quest to understand stable varieties in higher dimensions.

**OBSERVATION 2.3.** For a stable family of curves,  $X \rightarrow B$ , let  $\tilde{X} \rightarrow X$  be a resolution of singularities and  $0 \in B$  a point. Then

- (2.3.1)  $\omega_{X/B}$  is relatively ample;
- (2.3.2) the special fiber  $X_0$  is uniquely determined by the rest of the family;
- (2.3.3)  $X$  has Du Val singularities; and
- (2.3.4)  $\tilde{X} \rightarrow B$  has reduced fibers;

### 3. CANONICAL MODELS

Next we will investigate how stability may be generalized to higher dimensions. For a more detailed study and many other results see [KSB88].

First let us consider our goals. One wants to find a class of singularities that allows us to define a moduli functor that would compactify the moduli functor of smooth canonically polarized varieties. In other words, one wants to define *stable* varieties as canonically polarized varieties with singularities only from this

particular class and one would like that any family of smooth canonically polarized varieties over a punctured curve have a unique stable limit, possibly over a branched covering which is totally ramified over the punctured point.

Taking into account previous observations in the case of families of curves, this means that one would like to achieve a notion of stable families such that (2.3.1) and (2.3.2) remain true. The first of these conditions is simply saying that stable varieties should be canonically polarized. This is both reasonable and expected and if one is familiar with the construction of moduli spaces via the Hilbert scheme (see for instance [Vie95]) then one can see that this is also necessary for other reasons as well. The second condition, that is, uniqueness of specialization is important with regard to the moduli space one hopes to construct eventually: this condition is essentially saying that this moduli space would be separated, surely a condition one would like to have.

The other two conditions in (2.3), namely (2.3.3) and (2.3.4) are actually the ones that will help us figure out the right class of singularities having the desired properties mentioned above.

It turns out that (2.3.1) and (2.3.3) combined implies the uniqueness of specialization, that is, once one has (2.3.1), then (2.3.3) actually implies (2.3.2). The last condition, (2.3.4) will be useful in determining what class of singularities would the fibers need to have in order for the total space to have the kind of singularities that are the appropriate generalization of Du Val singularities in the case of families of curves. We will investigate this further in §4.

*Du Val singularities*, also known as *rational double points*, or *canonical Gorenstein surface singularities* may be defined a number of ways, see [Dur79] for fifteen of these. The original definition of them is actually the one that generalizes well to higher dimensions.

In the following I will need to use the canonical sheaf on singular varieties. If  $X$  is Cohen-Macaulay, then a dualizing sheaf exists and the canonical sheaf may be defined as that. For the definition in more general settings please see §5.

**DEFINITION 3.1.** Let  $X$  be a normal variety and assume that it admits a canonical sheaf  $\omega_X$  which is a line bundle. (This holds for example if  $X$  is Gorenstein). Then  $X$  has *canonical singularities* if for a resolution of singularities  $\phi : \tilde{X} \rightarrow X$  one has the following:

$$\phi^* \omega_X \subseteq \omega_{\tilde{X}}.$$

If  $\dim X = 2$ , these are also called *Du Val singularities*.

**REMARK 3.2.** The assumption that  $\omega_X$  is a line bundle is in fact not necessary to define canonical singularities, but it makes the definition simpler. We will later extend the definition to a larger class.

Notice that the (injective) morphism  $\phi^*\omega_X \rightarrow \omega_{\tilde{X}}$  does not always exist. However, if a non-zero morphism like that exists, then it is necessarily injective cf. (3.3)

Even though such a morphism does not always exist, it is easy to see that it does if  $X$  is smooth. Indeed, in that case there exists a natural morphism induced by the pull-back of differential forms  $\phi^*\Omega_X \rightarrow \Omega_{\tilde{X}}$  and taking determinants implies the existence of a non-zero morphism  $\phi^*\omega_X \rightarrow \omega_{\tilde{X}}$ .

**Lemma 3.3.** *Let  $Y$  be an irreducible variety,  $\mathcal{L}$  and  $\mathcal{F}$  torsion-free sheaves on  $Y$ , and  $\alpha : \mathcal{L} \rightarrow \mathcal{F}$  a non-zero morphism. If  $\mathcal{L}$  has rank 1, then  $\alpha$  must be injective.*

*Proof.* Let  $\mathcal{K} = \ker \alpha$  and  $\mathcal{I} = \operatorname{im} \alpha$ . If  $\alpha$  is non-zero, then  $\mathcal{I}$  is a non-zero subsheaf of the torsion-free  $\mathcal{F}$ . Since  $\mathcal{L}$  is rank 1 (at the general point of  $Y$ ) it follows that so is  $\mathcal{I}$ . Therefore  $\alpha$  is generically injective which implies that  $\mathcal{K}$  is a torsion sheaf. However,  $\mathcal{L}$  is also torsion-free and hence  $\mathcal{K} = 0$ .  $\square$

**Corollary 3.4.** *Let  $X$  be a normal variety and assume that it admits a canonical sheaf  $\omega_X$  which is a line bundle. If for a resolution of singularities  $\phi : \tilde{X} \rightarrow X$  there exists a non-zero morphism  $\phi^*\omega_X \rightarrow \omega_{\tilde{X}}$ , then  $X$  has canonical singularities. In particular, if  $X$  is smooth, then it has canonical singularities and in the definition of canonical singularities if the required condition holds for a single resolution of singularities, then it holds for all of them.*

*Proof.* Left to the reader.  $\square$

This leads us to another interesting condition that characterizes canonical singularities of Gorenstein varieties.

**Lemma 3.5.** *Let  $X$  be a normal variety and assume that it admits a canonical sheaf  $\omega_X$  which is a line bundle and  $\phi : \tilde{X} \rightarrow X$  a resolution of singularities. Then the following are equivalent:*

(3.5.1)  $X$  has canonical singularities;

(3.5.2)  $\phi_*\omega_{\tilde{X}} \simeq \omega_X$ ; and

(3.5.3)  $\phi_*\omega_{\tilde{X}}^{\otimes m} \simeq \omega_X^{\otimes m}$  for all  $m \geq 0$ .

*Proof.* First assume that  $\phi_*\omega_{\tilde{X}} \simeq \omega_X$ . Notice that there always exists a natural morphism  $\phi_*\omega_{\tilde{X}} \rightarrow \omega_X$ , which is injective by (3.3), so this condition could be phrased by saying that “the natural morphism  $\phi_*\omega_{\tilde{X}} \rightarrow \omega_X$  is surjective”. In fact, the point of the condition is that this isomorphism implies that there exists a non-zero morphism  $\omega_X \rightarrow \phi_*\omega_{\tilde{X}}$  and via adjointness of  $\phi^*$  and  $\phi_*$  that implies the existence of a non-zero morphism  $\phi^*\omega_X \rightarrow \omega_{\tilde{X}}$ , which in turn implies that  $X$  has canonical singularities.

Now assume that  $X$  has canonical singularities, that is, there exists an injective morphism  $\phi^*\omega_X \rightarrow \omega_{\tilde{X}}$ . It follows that the line bundle  $\omega_{\tilde{X}} \otimes \phi^*\omega_X^{-1}$  corresponds

to an effective Cartier divisor  $E$  on  $\tilde{X}$ , so one obtains the expression:

$$\omega_{\tilde{X}} \simeq \phi^* \omega_X \otimes \mathcal{O}_{\tilde{X}}(E).$$

Since  $X$  is normal it also follows that  $E$  is  $\phi$ -exceptional and hence

$$\omega_{\tilde{X}}|_E \simeq \mathcal{O}_E(E).$$

Therefore for any  $m \geq 0$  one has the following short exact sequence:

$$0 \longrightarrow \phi^* \omega_X^{\otimes m} \longrightarrow \omega_{\tilde{X}}^{\otimes m} \longrightarrow \mathcal{O}_{mE}(mE) \longrightarrow 0.$$

In order to finish the proof one needs to prove that  $\phi_* \mathcal{O}_{mE}(mE) = 0$ . This is easy to prove for surfaces, since the fact that  $E$  is exceptional implies that its self-intersection is negative, hence the sheaf  $\mathcal{O}_{mE}(mE)$  has no global sections. The statement in arbitrary dimension follows by a simple induction on the dimension considering general hyperplane sections. For details see [KMM87, 1-3-2].  $\square$

Combining canonical singularities with canonical polarization leads to the notion of *canonical models*:

**Theorem 3.6.** *Let  $X$  be a variety with canonical singularities and  $\phi : \tilde{X} \rightarrow X$  a resolution of singularities. Assume that  $\omega_X$  is ample. Then  $X$  is isomorphic to the canonical model of  $\tilde{X}$ . In particular one has that*

$$X \simeq \text{Proj} \bigoplus_{m \geq 0} H^0(\tilde{X}, \omega_{\tilde{X}}^{\otimes m}).$$

*Proof.* Since  $\omega_X$  is ample, it follows easily that

$$X \simeq \text{Proj} \bigoplus_{m \geq 0} H^0(X, \omega_X^{\otimes m}),$$

and  $H^0(X, \omega_X^{\otimes m}) \simeq H^0(\tilde{X}, \omega_{\tilde{X}}^{\otimes m})$  for any  $m \geq 0$  by (3.5.3).  $\square$

The same proof provides a relative version of this statement:

**Theorem 3.7.** *Let  $f : X \rightarrow B$  be a proper flat morphism and  $\phi : \tilde{X} \rightarrow X$  a resolution of singularities. Let  $\tilde{f} = f \circ \phi$  and assume that  $X$  has canonical singularities,  $B$  is a smooth curve and  $\omega_{X/B}$  is relatively ample with respect to  $f$ . Then one has a natural  $B$ -isomorphism*

$$X/B \simeq (\text{Proj}_B \bigoplus_{m \geq 0} \tilde{f}_* \omega_{\tilde{X}/B}^{\otimes m})/B.$$

*Proof.* Since  $\omega_{X/B}$  is relatively ample, it follows that

$$X/B \simeq (\text{Proj}_B \bigoplus_{m \geq 0} f_* \omega_{X/B}^{\otimes m})/B,$$

and  $f_* \omega_{X/B}^{\otimes m} \simeq f_* \omega_X^{\otimes m} \otimes \omega_B^{-m} \simeq \tilde{f}_* \omega_{\tilde{X}}^{\otimes m} \otimes \omega_B^{-m} \simeq \tilde{f}_* \omega_{\tilde{X}/B}^{\otimes m}$  for any  $m \geq 0$  by (3.5.3).  $\square$

**Corollary 3.8.** *Let  $B$  be a smooth curve,  $0 \in B$  a point, and  $B^\circ = B \setminus \{0\}$ . Let  $f : X \rightarrow B$  and  $f' : X' \rightarrow B$  be two proper flat morphisms such that restricting  $f$  and  $f'$  over  $B^\circ$  gives isomorphic families, i.e.,  $(X \times_B B^\circ)/B^\circ \simeq (X' \times_B B^\circ)/B^\circ$  as  $B^\circ$ -schemes. If both  $X$  and  $X'$  have canonical singularities and both  $\omega_{X/B}$  and  $\omega_{X'/B}$  are relatively ample, then  $X/B \simeq X'/B$  as  $B$ -schemes. In particular, the special fibers of  $f$  and  $f'$  are isomorphic:  $X_0 \simeq X'_0$ .*

*Proof.* Let  $\tilde{X}$  be a common resolution of singularities of  $X$  and  $X'$  with resolution morphisms be  $\phi : \tilde{X} \rightarrow X$  and  $\phi' : \tilde{X} \rightarrow X'$ . It follows that then  $f \circ \phi = f' \circ \phi'$  so one may denote this morphism by  $\tilde{f}$  and so

$$X/B \simeq (\mathrm{Proj}_B \bigoplus_{m \geq 0} \tilde{f}_* \omega_{\tilde{X}/B}^{\otimes m})/B \simeq X'/B$$

by (3.7). □

The important conclusion to draw from this is that in order to guarantee uniqueness of specialization one should require that a stable family has a relatively ample canonical sheaf and its total space has canonical singularities.

**OBSERVATION 3.9.** For a stable family  $X \rightarrow B$  over a smooth curve  $B$  let  $\tilde{X} \rightarrow X$  be a resolution of singularities and  $0 \in B$  a point. Then one expects the following conditions to hold:

- (3.9.1)  $\omega_{X/B}$  is relatively ample;
- (3.9.2)  $X$  has canonical singularities; and
- (3.9.3)  $\tilde{X} \rightarrow B$  has reduced fibers;

Notice that I dropped the condition that “the special fiber  $X_0$  is uniquely determined by the rest of the family” from (2.3) not because we no longer need it but because (3.9.1) and (3.9.2) imply it.

In the next section we will investigate what the third condition (3.9.3) gives us with regard to the singularities of the fibers.

#### 4. STABLE SINGULARITIES

Let  $f : X \rightarrow B$  be a flat morphism over a smooth curve  $B$ ,  $\phi : \tilde{X} \rightarrow X$  a resolution of singularities and  $0 \in B$  a point. Assume that  $X$  has canonical singularities and  $\tilde{f} : \tilde{X} \rightarrow B$  has reduced fibers.

One would like to understand the condition this places on the singularities of  $X_0$ , the special fiber of  $f$ . To this end let us assume that  $\phi$  is an embedded resolution of  $X_0 \subset X$  and such that  $\phi^* X_0 = \tilde{X}_0 \subset \tilde{X}$  is an snc divisor. Notice that by assumption  $B$  is a smooth curve, so  $X_0$  is a Cartier divisor and hence pulling it back makes sense. Furthermore, assume that  $\tilde{f}$  has reduced fibers so  $\phi^* X_0 = \tilde{X}_0$  itself is an snc divisor not just that its support is one.



We saw in the proof of (3.5) that  $X$  having canonical singularities implies, and in fact is equivalent to, that

$$(4.1) \quad \omega_{\tilde{X}} \simeq \phi^* \omega_X(E)$$

for some effective  $\phi$ -exceptional divisor  $E \subset \tilde{X}$ .

Since  $\phi$  is an embedded resolution of  $X_0 \subset X$ ,  $\tilde{X}_0$  contains a union of components  $\hat{X}_0$  that gives a resolution of singularities  $\hat{\phi}_0 = \phi|_{\hat{X}_0} : \hat{X}_0 \rightarrow X_0$ . One cannot, however, expect  $\tilde{X}_0$  be equal to  $\hat{X}_0$ , so one obtains that

$$(4.2) \quad \phi^* X_0 = \tilde{X}_0 = \hat{X}_0 + F,$$

where  $F$  is the effective  $\phi$ -exceptional divisor formed by the unions of the components of  $\tilde{X}_0$  not contained in  $\hat{X}_0$ . Since  $\tilde{X}$  is smooth, all of these are Cartier divisors.

By adjunction one has that  $\omega_{\hat{X}_0} \simeq \omega_{\tilde{X}}(\hat{X}_0)|_{\hat{X}_0}$  and  $\omega_{X_0} \simeq \omega_X(X_0)|_{X_0}$ . Combining this with (4.1) and (4.2) leads to the isomorphism

$$\begin{aligned} \omega_{\hat{X}_0} &\simeq \omega_{\tilde{X}}(\hat{X}_0)|_{\hat{X}_0} \simeq \phi^* \omega_X(E + \phi^* X_0 - F)|_{\hat{X}_0} \simeq \\ &\simeq \hat{\phi}_0^* \left( \omega_X(X_0)|_{X_0} \right) \otimes \mathcal{O}_{\hat{X}_0} \left( (E - F)|_{\hat{X}_0} \right) \simeq \hat{\phi}_0^* \omega_{X_0} \otimes \mathcal{O}_{\hat{X}_0} \left( (E - F)|_{\hat{X}_0} \right) \end{aligned}$$

Now let  $\hat{E}_0 = E|_{\hat{X}_0}$  and  $\hat{F}_0 = F|_{\hat{X}_0}$ . Then one obtains that

$$(4.3) \quad \hat{\phi}_0^* \omega_{X_0} \subseteq \omega_{\hat{X}_0}(\hat{F}_0)$$

This is not quite the definition of canonical singularities, but a somewhat weaker condition. Notice however that while we did not know much about the multiplicities of the components of  $E$  other than that they are non-negative, we do know that  $\tilde{X}_0 = \hat{X}_0 + F$  is an snc divisor and hence so is  $\hat{F}_0 = F \cap \hat{X}_0 \subset \hat{X}_0$ . This is an important detail. This means that although  $\hat{\phi}_0^* \omega_{X_0}$  does not necessarily admit a non-zero morphism to  $\omega_{\hat{X}_0}$ , it does admit an embedding to a slightly larger sheaf. This leads to the definition of *(semi) log canonical singularities* see §6 for more details.

Observe that the above computation works backwards as well, so we actually found what we were looking for: a condition on the singularities of the fibers instead of a condition on the singularities of the total space.

## 5. THE DUALIZING SHEAF VERSUS THE CANONICAL DIVISOR

In order to construct moduli spaces one needs a polarization of our objects. The (essentially only) natural choice of a line bundle on an abstract smooth projective variety is the canonical bundle. This is the main reason we are studying *canonically* polarized varieties. When one extends our moduli problem in order to have compact moduli spaces one still needs a canonical polarization. However, the dualizing sheaf, even if it exists, is not necessarily a line bundle. Therefore, a discussion of how one produces canonical polarizations on stable varieties is in

order. Below we will use many of the notions and notation from (1.1) but we also need a few more.

**DEFINITION 5.1.** A finitely generated non-zero module  $M$  over a noetherian local ring  $R$  is called *Cohen-Macaulay* if its depth over  $R$  is equal to its dimension. For the definition of depth and dimension I refer the reader to [BH93]. The ring  $R$  is called *Cohen-Macaulay* if it is a Cohen-Macaulay module over itself.

Let  $X$  be a scheme and  $x \in X$  a point. One says that  $X$  has *Cohen-Macaulay* singularities at  $x$  (or simply  $X$  is *CM* at  $x$ ), if the local ring  $\mathcal{O}_{X,x}$  is Cohen-Macaulay.

If in addition,  $X$  admits a dualizing sheaf  $\omega_X$  which is a line bundle in a neighbourhood of  $x$ , then  $X$  is *Gorenstein* at  $x$ .

The scheme  $X$  is *Cohen-Macaulay* (resp. *Gorenstein*) if it is *Cohen-Macaulay* (resp. *Gorenstein*) at  $x$  for all  $x \in X$ .

If  $X$  is Cohen-Macaulay, then it admits a dualizing sheaf. However, stable varieties are not necessarily Cohen-Macaulay, so one needs a more sophisticated approach.

Stable varieties are projective and projective varieties admit *dualizing complexes*: If  $X \subseteq \mathbb{P}^N$  and  $d = \dim X$ , then

$$\omega_X^\bullet \simeq_{\text{qis}} \mathcal{R}\mathcal{H}om_{\mathbb{P}^N}(\mathcal{O}_X, \omega_{\mathbb{P}^N}[N]).$$

Using this dualizing complex one can always define the *canonical sheaf*:

$$\omega_X := h^{-d}(\omega_X^\bullet)$$

In fact, this allows us to define the canonical sheaf of any quasi-projective variety, or more generally any locally closed subset of a variety that admits a dualizing complex. For more on this the reader is referred to [Har66, Con00].

Suppose  $U \subseteq X$  is an open subset of the projective variety  $X$ . Then let

$$\omega_U := \omega_X^\bullet|_U.$$

**REMARK 5.2.** Note that  $X$  is Cohen-Macaulay if and only if

$$\omega_X^\bullet \simeq_{\text{qis}} \omega_X[d],$$

that is, if the only non-zero cohomology sheaf of  $\omega_X^\bullet$  is the  $-d^{\text{th}}$  (and  $d$  still denotes  $\dim X$ ). In this case the canonical sheaf is isomorphic to the *dualizing sheaf*.

$X$  is Gorenstein if and only if it is Cohen-Macaulay and  $\omega_X$  is a line bundle.

For a normal variety  $X$  the usual way to define the canonical sheaf is different but produces the same sheaf. Being normal is equivalent to being  $R_1$  and  $S_2$ , that is,  $X$  is normal if and only if it is non-singular in codimension 1 and satisfies Serre's  $S_2$  condition.

Let  $U = X \setminus \text{Sing } X$  be the locus where  $X$  is non-singular and  $\iota : U \hookrightarrow X$  its natural embedding to  $X$ . Then one may define the canonical=dualizing sheaf of  $U$  as the determinant of the cotangent bundle, i.e., the sheaf of top differential

forms,  $\omega_U = \det \Omega_U$ . Then the usual definition of the canonical sheaf of  $X$  is  $\omega'_X := \iota_* \omega_U$ . It is relatively easy to see that both  $\omega_X$  and  $\omega'_X$  are reflexive and agree in codimension 1, so they are actually isomorphic (cf. [Har80, §1]).

$$\begin{array}{ccc} \omega_X & \xrightarrow{\simeq} & \omega'_X \\ \parallel & & \parallel \\ h^{-d}(\omega_X^\bullet) & & \iota_* \omega_U. \end{array}$$

Indeed, since  $\iota : U \hookrightarrow X$  is an open embedding, the restriction of the dualizing complex of  $X$  to  $U$  is the dualizing complex of  $U$ :

$$\omega_X^\bullet|_U \simeq_{\text{qis}} \omega_U^\bullet.$$

In particular, since restriction to  $U$  is an exact functor, one also has

$$\omega_X|_U \simeq \omega_U.$$

Recall that  $X$  is assumed to be normal. In that case the  $R_1$  condition implies that  $\text{codim}_X(X \setminus U) \geq 2$  and the  $S_2$  condition combined with the fact that  $\omega_X$  is reflexive implies that then

$$(5.3) \quad \omega_X \simeq \iota_* (\omega_X|_U) \simeq \iota_* \omega_U.$$

Possibly some readers are more familiar with this isomorphism in the divisor setting.

Let  $X$  be an irreducible normal variety and  $\iota : U \hookrightarrow X$  the non-singular locus as above. A *canonical divisor*  $K_X$  of  $X$  is a Weil divisor whose associated *Weil divisorial sheaf*,

$$\mathcal{O}_X(K_X) := \{f \in K(X) \mid K_X + \text{div}(f) \geq 0\},$$

is isomorphic to the canonical sheaf  $\omega_X$ . This is usually defined the following way: Define  $\omega_U$  as above. As  $U$  is non-singular,  $\omega_U$  is a line bundle and hence corresponds to a Cartier divisor. Let  $K_U = \sum \lambda_i K_i$  denote a Weil divisor associated to this Cartier divisor. Let  $\overline{K}_i$  denote the closure of  $K_i$  in  $X$  and let

$$K_X := \sum \lambda_i \overline{K}_i.$$

Since  $\text{codim}_X(X \setminus U) \geq 2$ , this is the unique Weil divisor on  $X$  for which  $K_X|_U = K_U$ . By the same argument as in the paragraph preceding (5.3) it follows that

$$\omega_X \simeq \mathcal{O}_X(K_X).$$

As already clear from the case of curves, when working with objects on the boundary of the moduli space one is forced to work with non-normal schemes. We will need one more important detail to make this work. Notice that  $U$  being non-singular is not essential in the above constructions. Since one knows how to define the canonical sheaf of a quasi-projective variety, one does not need  $U$  to be non-singular for that. The only place where we used the non-singularity of  $U$  was

to establish that  $\omega_U$  is a line bundle. In other words, we may replace the condition of  $U$  being non-singular with assuming that its canonical sheaf is a line bundle. In particular, assuming that  $U$  is Gorenstein will do the trick and then we still have that

$$(5.4) \quad \omega_X \simeq \iota_* (\omega_X|_U) \simeq \iota_* \omega_U.$$

The precise condition we need in order to be able to define stable varieties is the following.

**DEFINITION 5.5.** A variety is called  $G_1$  if it is Gorenstein in codimension 1.

If  $X$  is  $G_1$  and  $S_2$  then everything said about the canonical sheaf of normal varieties above works the same way. In particular, one may talk about a *canonical divisor*  $K_X$  which is a Weil divisor that is Cartier in codimension 1. In fact, if  $X$  is  $G_1$  and  $S_2$ , then one does not need to assume that  $X$  admits a dualizing complex and one does not need to define the canonical sheaf that way:

**DEFINITION 5.6.** Let  $X$  be a scheme that is  $G_1$  and  $S_2$  and  $\iota : U \hookrightarrow X$  be an open set such that  $\text{codim}_X(X \setminus U) \geq 2$  and  $U$  is Gorenstein. Then

$$\omega_X := \iota_* \omega_U$$

is called the *canonical sheaf* of  $X$ .

**Lemma 5.7.** *If  $X$  admits a dualizing complex, using the above definition for  $\omega_X$ , one still has that*

$$\omega_X \simeq h^{-d}(\omega_X^\bullet),$$

where  $d = \dim X$ . In particular, the two definitions of the dualizing sheaf agree.

*Proof.* This follows from (5.4). □

**REMARK 5.8.** We are now in a perfect position to take a deep breath, make a few observations, and lose any inhibition we might have against working with non-normal varieties. Being normal is the same as being  $R_1$  and  $S_2$  and we are replacing that with being  $G_1$  and  $S_2$ . In other words, we are not going wild with all kinds of weird schemes. As far as our canonical divisors are concerned we are not much worse off than working with normal varieties. The main thing to keep in mind is that our varieties may be singular along a divisor. This means that for example one has to be careful when working with Weil divisors. However, the extent of this is essentially that by the  $G_1$  assumption  $\omega_X$  is a line bundle near the general points of the 1-codimensional part of the singular locus of  $X$  and hence we may choose canonical divisors whose support does not contain any components of that 1-codimensional singular locus. This implies that  $X$  is non-singular at the general points of these canonical divisors, so we may work with them as we are used to work with Weil divisors. In addition, we will put even more restrictions on our singularities. In particular, our stable varieties will only

have double normal crossssings in codimension 1. These are arguably the simplest non-normal singularities and they are also Gorenstein.

As indicated at the beginning of this section, in order to construct our moduli spaces one needs a canonical polarization on our stable varieties. The obvious assumption would be to require that stable varieties are Gorenstein. This works in dimension 1, but not in higher dimension. Consider a cone over a quartic rational scroll in  $\mathbb{P}^5$ . Then a general pencil of hyperplanes defines a family of smooth varieties degenerating to one that is not Gorenstein; a cone over a quartic rational curve in  $\mathbb{P}^4$ . For a more detailed explanation of this example see [A](#). Taking a branched cover over a general high degree hypersurface section of the cone one obtains a family of smooth canonically polarized varieties degenerating to one with the same kind of singularities as above. This example shows that if one sticks to Gorenstein singularities, or even just to those for which  $\omega_X$  is a line bundle, one will not get a compact moduli space.

So, if  $\omega_X$  is not a line bundle, how does one get a “canonical polarization”? The point is that even though one cannot assume that  $\omega_X$  is a line bundle, may assume that some power of it is. Of course, since  $\omega_X$  is not a line bundle, one has to be careful what “power” means. Tensor powers of non-locally free sheaves tend to get even worse. For instance, tensor powers of torsion-free, or even reflexive sheaves may have torsion or co-torsion. Also, we want the power to be still associated to a Weil divisor. In other words, we want it to be a reflexive sheaf, i.e., we need to take reflexive powers:

**DEFINITION 5.9.** Let  $X$  be a scheme that admits a canonical sheaf  $\omega_X$ . (For instance it admits a dualizing complex or it is  $G_1$  and  $S_2$ ). Then one defines the *pluricanonical sheaves* of  $X$  as the reflexive powers of the canonical sheaf of  $X$ :

$$\omega_X^{[m]} := (\omega_X^{\otimes m})^{**}.$$

**Lemma 5.10.** *Let  $X$  be a scheme that is  $G_1$  and  $S_2$ . Then for any  $m \in \mathbb{Z}$ ,*

$$\omega_X^{[m]} \simeq \mathcal{O}_X(mK_X).$$

*Proof.* Let  $\iota : U \hookrightarrow X$  be an open dense subset of  $X$  such that  $\text{codim}_X(X \setminus U) \geq 2$  and  $\omega_X|_U \simeq \omega_U$  is a line bundle. It follows that  $\omega_X^{\otimes m}|_U \simeq \omega_U^{\otimes m}$  is a line bundle, and hence

$$\omega_X^{[m]}|_U \simeq \mathcal{O}_X(mK_X)|_U.$$

Since both  $\omega_X^{[m]}$  and  $\mathcal{O}_X(mK_X)$  are reflexive, this means that they are isomorphic cf. [\[Har94, 1.11\]](#).  $\square$

This means that if  $X$  is  $G_1$  and  $S_2$ , then one may work with pluricanonical divisors the same way as if  $X$  was normal.

**REMARK 5.11.** Talking about Weil divisors on non-normal schemes is tricky, because in order to define the multiplicity of a function along a prime divisor and

hence define the notion of linear equivalence of Weil divisors, one needs the local rings of general points of these prime divisors to be DVRs. Therefore, one only considers prime divisors that are not contained in the singular locus of the ambient scheme. The condition  $G_1$  ensures that the canonical sheaf may be represented by a Weil divisor that satisfies this requirement.

We are now ready to introduce the notion that allows us to have canonical polarizations even if  $\omega_X$  is not a line bundle.

**DEFINITION 5.12.** Let  $X$  be a scheme that admits a canonical sheaf  $\omega_X$ . Then, as in (1.1),  $\omega_X$  is called a  $\mathbb{Q}$ -line bundle if some pluricanonical sheaf  $\omega_X^{[m]}$  is a line bundle.

As a direct consequence of (5.10) one obtains:

**Lemma 5.13.** *Let  $X$  be a scheme that is  $G_1$  and  $S_2$ . Then  $K_X$  is  $\mathbb{Q}$ -Cartier if and only  $\omega_X$  is a  $\mathbb{Q}$ -line bundle.*

## 6. SINGULARITIES OF THE MINIMAL MODEL PROGRAM

It is time to take a more detailed look at the singularities we have encountered and give precise definitions. For an excellent introduction to this topic the reader is urged to take a thorough look at Miles Reid's *Young person's guide to canonical singularities* [Rei87]. For the precise theory the standard reference is [KM98] and for recent results one may consult [HK10].

### 6.A. Log canonical singularities

As we have already seen in the case of stable curves, in order to construct compact moduli spaces one must deal with non-normal singularities as that is the nature of degenerations: normalization does not work in families. However, as a warm-up, let us first define the normal and more traditional singularities that are relevant in the minimal model program. This will help understanding the somewhat more technical definitions required to deal with the non-normal case.

**DEFINITION 6.1.** Let  $X$  be a normal variety such that  $K_X$  is  $\mathbb{Q}$ -Cartier and  $\phi : \tilde{X} \rightarrow X$  a resolution of singularities with a normal crossing exceptional divisor  $E = \cup E_i$ . One would like to compare the canonical divisors of  $\tilde{X}$  and  $X$ . Since  $\phi$  is an isomorphism on an open set this means that the relative canonical divisor, that is, the difference between  $K_{\tilde{X}}$  and the pull-back of  $K_X$  is a divisor supported entirely on the exceptional locus. However, as  $K_X$  is not necessarily Cartier one may not be able to pull it back. One may pull back a multiple of it, so one compares that to the same multiple of  $K_{\tilde{X}}$ . Then one divides the difference by the appropriate power. Notice that this way one may actually define the pull-back of  $K_X$  as a  $\mathbb{Q}$ -divisor:

$$\phi^* K_X := \frac{1}{m} \phi^*(mK_X),$$

where  $m$  is such that  $mK_X$  is Cartier. Then one may indeed compare the canonical divisors of  $\tilde{X}$  and  $X$ :

$$K_{\tilde{X}} \sim_{\mathbb{Q}} \phi^* K_X + \sum a_i E_i.$$

where  $a_i \in \mathbb{Q}$ . Then

$$X \text{ has } \begin{array}{l} \textit{terminal} \\ \textit{canonical} \\ \textit{log terminal} \\ \textit{log canonical} \end{array} \quad \text{singularities if } \begin{array}{l} a_i > 0. \\ a_i \geq 0. \\ a_i > -1. \\ a_i \geq -1. \end{array}$$

for all  $i$  and any resolution  $\phi$  as above.

REMARK 6.2. We saw in §4 that the “right” class of singularities for the total space of a stable family is that of *canonical singularities* and that this leads to the fibers having *log canonical singularities*. Here we extended the definition of canonical singularities from varieties whose canonical sheaf is a line bundle to those whose canonical sheaf is a  $\mathbb{Q}$ -line bundle. We will generalize these definitions to include the non-normal relatives of these singularities in §§6.D which will be the right class for “stable singularities”.

Next we will see further evidence supporting this claim.

EXAMPLE 6.3. This is an auxiliary example that I will use later.

Let  $\Xi = (x^d + y^d + z^d + tw^d = 0) \subseteq \mathbb{P}_{x:y:z:w}^3 \times \mathbb{A}_t^1$ . The special fiber  $\Xi_0$  is a cone over a smooth plane curve of degree  $d$  and the general fiber  $\Xi_t$ , for  $t \neq 0$ , is a smooth surface of degree  $d$  in  $\mathbb{P}^3$ .

FACT 6.4. Let  $W$  be a smooth variety and  $X = X_1 \cup X_2 \subseteq W$  such that  $X_1$  and  $X_2$  are Cartier divisors in  $W$ . Then by adjunction

$$\begin{aligned} K_X &\sim (K_W + X_1 + X_2)|_X \\ K_{X_1} &\sim (K_W + X_1)|_{X_1} \\ K_{X_2} &\sim (K_W + X_2)|_{X_2}, \end{aligned}$$

and hence

$$\begin{aligned} K_X|_{X_1} &\sim K_{X_1} + X_2|_{X_1} \\ K_X|_{X_2} &\sim K_{X_2} + X_1|_{X_2}. \end{aligned}$$

It turns out that these latter equalities are true under more general conditions and hence they allow one to check when the canonical divisor of a reducible variety is ample by working with the canonical divisor of its irreducible components.

EXAMPLE 6.5. As before, let  $f : X \rightarrow B$  be a flat morphism,  $B$  a smooth curve, and  $\phi : \tilde{X} \rightarrow X$  a resolution of singularities. In this example assume that  $X_0$  is a normal projective surface with  $K_{X_0}$  ample and an isolated singular point  $P \in \text{Sing } X_0$  such that  $X_0$  is isomorphic to a cone  $\Xi_0 \subseteq \mathbb{P}^3$  as in Example 6.3 locally analytically near  $P$ . Assume further that  $X$  is smooth. One would like to see whether one may resolve the singular point  $P \in X_0$  and still stay within

our moduli problem, i.e., that  $K$  would remain ample. For this purpose one may assume that  $P$  is the only singular point of  $X_0$ .

Because of the assumption on the singularities one may assume that  $\phi$  is the blowing up of  $P \in X$  and let  $\hat{X}_0$  denote the strict transform of  $X_0$  on  $\tilde{X}$ . Then  $\tilde{X}_0 = \hat{X}_0 \cup E$  where  $E \simeq \mathbb{P}^2$  is the exceptional divisor of the blow up. Clearly,  $\phi : \hat{X}_0 \rightarrow X_0$  is the blow up of  $P$  on  $X_0$ , so it is a smooth surface and  $\hat{X}_0 \cap E$  is isomorphic to the degree  $d$  curve over which  $X$  is locally analytically a cone.

One would like to determine the condition on  $d$  that ensures that the canonical divisor of  $\tilde{X}_0$  is still ample. According to (6.4) this means that one needs that  $K_E + \hat{X}_0|_E$  and  $K_{\hat{X}_0} + E|_{\hat{X}_0}$  be ample. As  $E \simeq \mathbb{P}^2$ ,  $\omega_E \simeq \mathcal{O}_{\mathbb{P}^2}(-3)$ , so  $\mathcal{O}_E(K_E + \hat{X}_0|_E) \simeq \mathcal{O}_{\mathbb{P}^2}(d-3)$ . This is ample if and only if  $d > 3$ .

As this computation is local near  $P$  the only relevant issue about the ampleness of  $K_{\hat{X}_0} + E|_{\hat{X}_0}$  is whether it is ample in a neighbourhood of  $E_0 := E|_{\hat{X}_0}$ . By (6.6) this is equivalent to asking when  $(K_{\hat{X}_0} + E_0) \cdot E_0$  is positive.

**CLAIM 6.6.** Let  $Z$  be a smooth projective surface with non-negative Kodaira dimension and  $\Gamma \subset Z$  an effective divisor. If  $(K_Z + \Gamma) \cdot C > 0$  for every proper curve  $C \subset Z$ , then  $K_Z + \Gamma$  is ample.

*Proof.* By the assumption on the Kodaira dimension there exists an  $m > 0$  such that  $mK_Z$  is effective, hence so is  $m(K_Z + \Gamma)$ . Then by the assumption on the intersection number,  $(K_Z + \Gamma)^2 > 0$ , so the statement follows by the Nakai-Moishezon criterium.  $\square$

Now, observe that by the adjunction formula  $(K_{\hat{X}_0} + E_0) \cdot E_0 = \deg K_{E_0} = d(d-3)$  as  $E_0$  is isomorphic to a plane curve of degree  $d$ . Again, one obtains the same condition as above and thus conclude that  $K_{\hat{X}_0}$  is ample if and only if  $d > 3$ .

Since the objects that one considers in the current moduli problem must have an ample canonical class, one may only replace  $X_0$  by  $\tilde{X}_0$  if  $d > 3$ . For our moduli problem this means that one has to allow cone singularities over curves of degree  $d \leq 3$ . The singularity one obtains for  $d = 2$  is a rational double point, but the singularity for  $d = 3$  is not, it is not even rational.

In fact, the above calculation tells us more. One has that  $K_{\hat{X}_0} = \phi^*K_{X_0} + aE_0$  for some  $a \in \mathbb{Z}$ . To compute  $a$ , first recall that  $\deg K_{E_0} = d(d-3)$  and  $E_0^2 = -d$ . Then

$$\deg K_{E_0} = (K_{\hat{X}_0} + E_0) \cdot E_0 = (\phi^*K_{X_0} + (a+1)E_0) \cdot E_0 = (a+1)E_0^2 = -(a+1)d.$$

Therefore  $a = 2 - d$ . In other words, the condition obtained above, that one needs to allow cone singularities over plane curves of degree  $d \leq 3$  is equivalent to allowing log canonical singularities cf. (6.1).

I have mentioned that stable singularities are not necessarily Cohen-Macaulay. Until we identified the actual class we want to call stable this was more or less an empty statement. By now, it is rather clear that log canonical singularities will



belong to the class we are looking for, so we might as well point to an example of non-CM log canonical singularities.

EXAMPLE 6.7. Let  $X$  be a cone over an abelian variety of dimension at least 2. Then  $X$  is log canonical, but not Cohen-Macaulay.

As mentioned several times, one also has to deal with some non-normal singularities and in fact in the example in (6.5) one does not really need that  $X$  be normal. In the next few subsections we will see examples of non-normal singularities that one has to handle. In particular, we will see that one has to allow the non-normal cousins of log canonical singularities. These are called *semi-log canonical* singularities and the reader can find their definition in (6.D).

### 6.B. Normal crossings

A *normal crossing* singularity is one that is locally analytically (or formally) isomorphic to the intersection of coordinate hyperplanes in a linear space. In other words, it is a singularity locally analytically defined as  $(x_1x_2 \cdots x_r = 0) \subseteq \mathbb{A}^n$  for some  $r \leq n$ . In particular, as opposed to the curve case, for surfaces it allows for triple intersections. However, triple (or higher) intersections may be “semi-resolved”: Let  $X = (xyz = 0) \subseteq \mathbb{A}^3$ . Blow up the origin  $O \in \mathbb{A}^3$ ,  $\sigma : \text{Bl}_O \mathbb{A}^3 \rightarrow \mathbb{A}^3$  and consider the strict transform of  $X$ ,  $\sigma : \tilde{X} \rightarrow X$ . Observe that  $\tilde{X}$  has only double normal crossings and the morphism  $\sigma$  is an isomorphism over  $X \setminus \{O\}$ . Therefore, this is a semi-resolution as defined in (6.11.4). Double normal crossings cannot be resolved the same way, because the double locus is of codimension 1, so any morphism from any space with any kind of singularities that are not double normal crossings would fail to be an isomorphism in codimension 1.

Since normal crossings are (analytically) locally defined by a single equation, they are Gorenstein and hence the canonical sheaf  $\omega_X$  is still a line bundle and so it makes sense to require it to be ample.

These singularities already appear for stable curves, so it is not surprising that they are still here. As one wants to understand degenerations of one’s preferred families, one has to allow (at least) normal crossings.

Another important point to remember about normal crossings is that they are *not* normal. For some interesting and perhaps surprising examples of surfaces with normal crossings see [Kol07].

### 6.C. Pinch points

Another non-normal singularity that can occur as the limit of smooth varieties is the *pinch point*. It is locally analytically defined as  $(x_1^2 = x_2x_3^2) \subseteq \mathbb{A}^n$  ( $n \geq 3$ ). This singularity is a double normal crossing away from the pinch point. Its normalization is smooth, but blowing up the pinch point does not make it any better as shown by the example that follows.

EXAMPLE 6.8. Let  $X = (x_1^2 = x_2^2 x_3) \subseteq \mathbb{A}^3$ , where  $x_1, x_2, x_3$  are linear coordinates on  $\mathbb{A}^3$ ,  $O = (0, 0, 0)$  and compute  $\text{Bl}_O X$ . First, recall that

$$\text{Bl}_O \mathbb{A}^3 = \{(x_1, x_2, x_3) \times [y_1 : y_2 : y_3] \mid x_i y_j = x_j y_i \text{ for } i, j = 1, 2, 3\} \subset \mathbb{A}^3 \times \mathbb{P}^2,$$

where  $y_1, y_2, y_3$  are homogenous coordinates on  $\mathbb{P}^2$ .

- (6.8.1) Assume that  $y_1 = 1$ . Then  $x_2 = x_1 y_2$  and  $x_3 = x_1 y_3$  and the equation of the preimage of  $X$  becomes  $x_1^2 = x_1^3 y_2^2 y_3$ . This breaks up into  $x_1^2 = 0$  and  $1 = x_1 y_2^2 y_3$ . The former equation defines the exceptional divisor and the latter defines the strict transform of  $X$ , i.e.,  $\text{Bl}_O X$ . This does not have any points over  $O \in X$ , so on this chart, the blow up morphism  $\text{Bl}_O X \rightarrow X$  is an isomorphism and  $\text{Bl}_O X$  is smooth.
- (6.8.2) Assume that  $y_2 = 1$ . Then  $x_1 = x_2 y_1$  and  $x_3 = x_2 y_3$  and the equation of the preimage of  $X$  becomes  $x_2^2 y_1^2 = x_2^3 y_3$ . This breaks up into  $x_2^2 = 0$  and  $y_1^2 = x_2 y_3$ . Again, the former equation defines the exceptional divisor and the latter the strict transform of  $X$ ,  $\text{Bl}_O X$ . Notice that on this chart a coordinate system is given by  $x_2, y_1, y_3$  and the equation defines a quadric cone. Then blowing up the vertex of the cone gives a resolution on this chart.
- (6.8.3) Assume that  $y_3 = 1$ . Then  $x_1 = x_3 y_1$  and  $x_2 = x_3 y_2$  and the equation of the preimage of  $X$  becomes  $x_3^2 y_1^2 = x_3^3 y_2^2$ . This breaks up as  $x_3^2 = 0$  and  $y_1^2 = y_2^2 x_3$ . Again, the former equation defines the exceptional divisor and the latter the strict transform of  $X$ ,  $\text{Bl}_O X$ . Notice that on this chart a coordinate system is given by  $x_3, y_1, y_3$  and the latter equation is the same as the one we started with. So,  $\text{Bl}_O X$  again has a pinch point.

This computation shows that the blow-up of a pinch point will be, if anything, more singular, than the original and at best it can be resolved to be a pinch point again.

From this example one concludes that a pinch point cannot be resolved or even just made somewhat “better” by only trying to change it over the pinch point. It may only be resolved by taking the normalization. As in the case of double normal crossings, this is not an isomorphism in codimension 1.

OBSERVATION 6.9. Double normal crossings and pinch points share the following interesting properties:

- (6.9.1) Their normalization is smooth.
- (6.9.2) The normalization morphism is *not* an isomorphism in codimension 1.
- (6.9.3) It is not possible to find a partial resolution that is an isomorphism in codimension 1 that would make them better in any reasonable sense.

REMARK 6.10. Notice that all normal crossings share the first two properties, but, in dimension at least 2, not the third one as they may be partially resolved to double normal crossings.

One concludes that double normal crossing and pinch point singularities are unavoidable. However, at the same time, they should be viewed as the simplest non-normal singularities. In fact, in some sense they are much simpler than most normal singularities.

Furthermore, all other singularities can be resolved to these: Any reduced scheme admits a partial resolution to a scheme with only double normal crossings and pinch points such that the resolution morphism is an isomorphism wherever the original scheme is smooth, or has only double normal crossings or pinch points [Kol08b]. Of course, this only gives a partial resolution that is an isomorphism in codimension 1 if the scheme one starts with has double normal crossings in codimension 1 already. However, this turns out to be a condition one can achieve.

We will discuss relevant partial resolutions in more detail in (6.11).

#### 6.D. Semi-log canonical singularities

Next, I will make the definition of the non-normal version of log canonical singularities precise.

DEFINITION 6.11. Let  $X$  be a scheme of dimension  $n$  and  $x \in X$  a closed point.

(6.11.1)  $x \in X$  is a *double normal crossing* if it is locally analytically (or formally) isomorphic to the singularity

$$\{0 \in (x_0x_1 = 0)\} \subseteq \{0 \in \mathbb{A}^{n+1}\},$$

where  $n \geq 1$ .

(6.11.2)  $x \in X$  is a *pinch point* if it is locally analytically (or formally) isomorphic to the singularity

$$\{0 \in (x_0^2 = x_1^2x_2)\} \subseteq \{0 \in \mathbb{A}^{n+1}\},$$

where  $n \geq 2$ .

(6.11.3)  $X$  is *semi-smooth* if all closed points of  $X$  are either smooth, or a double normal crossing, or a pinch point. In this case, unless  $X$  is smooth,  $D_X := \text{Sing } X \subseteq X$  is a smooth  $(n-1)$ -fold. If  $\nu : \tilde{X} \rightarrow X$  is the normalization, then  $\tilde{X}$  is smooth and  $\tilde{D}_X := \nu^{-1}(D_X) \rightarrow D_X$  is a double cover ramified along the pinch locus. Furthermore, the definition implies that if  $X$  is semi-smooth, then it is Gorenstein. In particular, it admits a canonical sheaf  $\omega_X$  which is a line bundle.

(6.11.4) A morphism,  $\phi : Y \rightarrow X$  is a *semi-resolution* if

- $\phi$  is proper,
- $Y$  is semi-smooth,
- no component of  $D_Y$  is  $\phi$ -exceptional, and
- there exists a closed subset  $Z \subseteq X$ , with  $\text{codim}(Z, X) \geq 2$  such that

$$\phi|_{\phi^{-1}(X \setminus Z)} : \phi^{-1}(X \setminus Z) \xrightarrow{\sim} X \setminus Z$$

is an isomorphism.

Let  $E$  denote the exceptional divisor (i.e., the codimension 1 part of the exceptional set, not necessarily the whole exceptional set) of  $\phi$ . Then  $\phi$  is a *good semi-resolution* if  $E \cup D_Y$  is a divisor with global normal crossings on  $Y$ .

- (6.11.5)  $X$  has *semi-log canonical* (slc) (resp. *semi-log terminal* (slt)) singularities if
- (a)  $X$  is reduced,
  - (b)  $X$  is  $S_2$ ,
  - (c)  $X$  admits a canonical sheaf  $\omega_X$ , which is a  $\mathbb{Q}$ -line bundle of index  $m$ , and
  - (d) there exists a good semi-resolution of singularities  $\phi : \tilde{X} \rightarrow X$  with exceptional divisor  $E = \cup E_i$  such that  $\omega_{\tilde{X}}^m \simeq \phi^* \omega_X^{[m]} \otimes \mathcal{O}_{\tilde{X}}(m \cdot \sum a_i E_i)$  with  $a_i \in \mathbb{Q}$  and  $a_i \geq -1$  (resp.  $a_i > -1$ ) for all  $i$ .

REMARK 6.12. A semi-smooth scheme has at worst hypersurface singularities, so in particular it is Gorenstein. This means that condition (6.11.5d) implies that  $X$  is  $G_1$ . In other words it follows that  $X$  admits a canonical sheaf. However, (6.11.5d) cannot be stated without assuming this first. On the other hand, it means that one may assume that  $X$  is  $G_1$  instead. In other words, without loss of generality one may define slc (resp. slt) singularities as those satisfying that

- (6.12.1)  $X$  is reduced,
  - (6.12.2)  $X$  is  $G_1$  and  $S_2$ ,
  - (6.12.3)  $\omega_X$  is a  $\mathbb{Q}$ -line bundle of index  $m$ , and
  - (6.12.4) there exists a good semi-resolution of singularities  $\phi : \tilde{X} \rightarrow X$  with exceptional divisor  $E = \cup E_i$  such that  $\omega_{\tilde{X}}^m \simeq \phi^* \omega_X^{[m]} \otimes \mathcal{O}_{\tilde{X}}(m \cdot \sum a_i E_i)$  with  $a_i \in \mathbb{Q}$  and  $a_i \geq -1$  (resp.  $a_i > -1$ ) for all  $i$ .
- (6.13) Furthermore, once one assumes that  $X$  is  $G_1$  and  $S_2$ , one may work with canonical divisors instead of canonical sheaves. In other words, we may also define slc (resp. slt) singularities as those satisfying that

- (6.13.1)  $X$  is reduced,
- (6.13.2)  $X$  is  $G_1$  and  $S_2$ ,
- (6.13.3)  $K_X$  is  $\mathbb{Q}$ -Cartier, and
- (6.13.4) there exists a good semi-resolution of singularities  $\phi : \tilde{X} \rightarrow X$  with exceptional divisor  $E = \cup E_i$  such that  $K_{\tilde{X}} \equiv \phi^* K_X + \sum a_i E_i$  with  $a_i \in \mathbb{Q}$  and  $a_i \geq -1$  (resp.  $a_i > -1$ ) for all  $i$ .

Again, (6.11.4) implies that  $X$  is  $G_1$ , but one needs that assumption even to work with  $K_X$ . Of course, instead of  $G_1$ , one may start by assuming that  $X$  admits a semi-resolution, then conclude that canonical divisors may be defined and then go on with the definition.

REMARK 6.14. It is relatively easy to prove that if  $X$  has semi-log canonical (resp. semi-log terminal) singularities, then the condition in (6.11.5d) follows for *all* good semi-resolutions.

REMARK 6.15. One may further generalize the notion of semi log canonical and define *weakly semi log canonical* singularities as those that are seminormal,  $S_2$  and with an appropriately chosen divisor on the normalization, that pair is log canonical. In this context semi-log canonical singularities are exactly those weakly semi-log canonical divisors that are  $G_1$ . For the precise definition and more details on these singularities and their relationships see [KSS10].

REMARK 6.16. In the definition of a semi-resolution, one could choose to require that the exceptional set be a divisor. This leads to slightly different notions. It is still to be seen whether this variation leads to anything interesting (that is, anything interesting that is different from all the interesting things the definition above leads to). For more on singularities related to semi-resolutions see [KSB88], [Kol92], and [Kol08b].

Now we are ready to define stable varieties in arbitrary dimensions.

DEFINITION 6.17. A variety  $X$  is called *stable* if

- (6.17.1)  $X$  is projective,
- (6.17.2)  $X$  has semi log canonical singularities, and
- (6.17.3)  $\omega_X$  is an ample  $\mathbb{Q}$ -line bundle.

REMARK 6.18. Notice that if  $\dim X = 1$ , then this is equivalent with the previous definition of a stable curve (2.1).

We should also revisit the definition of *stable families*. As opposed to the case of curves, our stable varieties are canonically polarized by a  $\mathbb{Q}$ -line bundle and not a line bundle. As far as embedding into a projective space, computing intersection numbers, and pulling back pluricanonical sheaves are concerned this does not cause a big difference. However it introduces an additional element to which one has to pay attention when dealing with families.

We do not simply want a family of canonically polarized varieties but a family where these canonical polarizations are compatible. In other words, we want a relative canonical polarization of the family that restricts to the canonical polarization of the members of the family. In particular, we want that for a stable family  $X \rightarrow B$ ,

$$(6.19) \quad \omega_{X/B}|_{X_b} \simeq \omega_{X_b} \quad \text{for all } b \in B.$$

It turns out that for curves this follows from the other assumptions and as a matter of fact we have also (secretly) assumed it during our quest for stable varieties cf. (3.9).

The only point to keep in mind is that now if one wants to define stable families only using properties of the fibers, as in the case of curves, then one might lose this condition accidentally. For an example that this can actually happen, that is, that there exists families of stable varieties that are not stable families in the sense of our earlier requirements see [A](#).

DEFINITION 6.20. A morphism  $f : X \rightarrow B$  is called a *weakly stable family* if it satisfies the following conditions:

- (6.20.1)  $f$  is flat and projective
- (6.20.2)  $\omega_{X/B}$  is a relatively ample  $\mathbb{Q}$ -line bundle
- (6.20.3)  $X_b$  has semi log canonical singularities for all  $b \in B$ .

This definition actually still hides one very important detail. The fact that  $\omega_{X/B}$  is a  $\mathbb{Q}$ -line bundle means that it has an index, that is, an integer  $N \in \mathbb{N}$  such that  $\omega_{X/B}^{[N]}$  is a line bundle and this is the smallest positive reflexive power of  $\omega_{X/B}$  which is a line bundle. It follows that then (cf. [\[HK04, 2.6\]](#)),

$$(6.21) \quad \omega_{X/B}^{[N]}|_{X_b} \simeq \omega_{X_b}^{[N]}.$$

In particular,  $\omega_{X_b}$  is a  $\mathbb{Q}$ -line bundle of index  $m$  for some  $m$  that divides  $N$ . This means that  $X_b$  may appear in weakly stable families whose relative canonical sheaf is a  $\mathbb{Q}$ -line bundle of index  $N$  for any multiple of  $m$ . This actually leads to a problem with respect to the moduli spaces of these families. There may be weakly stable families all of whose members have canonical sheaves of index  $m$ , but the relative canonical sheaf of the family has index  $N > m$ . In other words one might encounter families that are admissible as families of varieties of index  $N$  but not as families of varieties of index  $m$ , even though all members have index  $m$ . A reasonable resolution of this problem is to ask that besides (6.21) a similar restriction should hold for all reflexive powers of the relative canonical sheaf.

DEFINITION 6.22. A weakly stable family  $f : X \rightarrow B$  is called a *stable family* if it satisfies *Kollár's condition*, that is, for any  $m \in \mathbb{N}$

$$\omega_{X/B}^{[m]}|_{X_b} \simeq \omega_{X_b}^{[m]}.$$

REMARK 6.23. Notice that it is always true that the double dual of the restriction of the relative pluricanonical sheaf is the corresponding pluricanonical sheaf of the fiber:

$$\left(\omega_{X/B}^{[m]}|_{X_b}\right)^{**} \simeq \omega_{X_b}^{[m]},$$

so the main content of Kollár's condition is that the restriction of all pluricanonical sheaves have to be reflexive. For more on the definition of stable families and the corresponding moduli functors see [\[Kov09, §7\]](#).

REMARK 6.24. Notice further that Kollár's condition includes condition (6.19). Interestingly, it is not obvious that even this simple condition holds for weakly stable families. It holds for families of curves since stable curves are Gorenstein.

It also holds for families of surfaces since stable surfaces are Cohen-Macaulay on account of being  $S_2$  and this condition holds for families of Cohen-Macaulay varieties cf. [Con00, 3.5.1].

However, stable varieties of dimension  $\geq 3$  are not necessarily Cohen-Macaulay (6.7), so it is absolutely not obvious whether the relative canonical sheaf is invariant under base change. It turns out that this is actually true by (11.3) cf. [KK10]. To see that this invariance under base change for weakly stable families is highly non-trivial the reader is referred to the examples in [Pat10] that show that this statement is sharp in some reasonable sense.

## 7. DUALITY AND VANISHING

In this section I will first state two fundamental theorems that will be used later and then list a few vanishing theorems that are important in both the minimal model program and higher dimensional moduli theory.

Before anything else, we need a few definitions.

**DEFINITION 7.1.** Let  $X$  be a complex scheme (i.e., a scheme of finite type over  $\mathbb{C}$ ) of dimension  $n$ . Let  $D_{\text{filt}}(X)$  denote the derived category of filtered complexes of  $\mathcal{O}_X$ -modules with differentials of order  $\leq 1$  and  $D_{\text{filt},\text{coh}}(X)$  the subcategory of  $D_{\text{filt}}(X)$  of complexes  $K$ , such that for all  $i$ , the cohomology sheaves of  $Gr_{\text{filt}}^i K$  are coherent cf. [DB81], [GNPP88]. Let  $D(X)$  and  $D_{\text{coh}}(X)$  denote the derived categories with the same definition except that the complexes are assumed to have the trivial filtration. The superscripts  $+$ ,  $-$ ,  $b$  carry the usual meaning (bounded below, bounded above, bounded). Isomorphism in these categories is denoted by  $\simeq_{\text{qis}}$ . A sheaf  $\mathcal{F}$  is also considered as a complex  $\mathcal{F}^\bullet$  with  $\mathcal{F}^0 = \mathcal{F}$  and  $\mathcal{F}^i = 0$  for  $i \neq 0$ . If  $K$  is a complex in any of the above categories, then  $h^i(K)$  denotes the  $i$ -th cohomology sheaf of  $K$ .

The right derived functor of an additive functor  $F$ , if it exists, is denoted by  $\mathcal{R}F$  and  $\mathcal{R}^i F$  is short for  $h^i \circ \mathcal{R}F$ . Furthermore,  $\mathbb{H}^i$  will denote  $\mathcal{R}^i \Gamma$ , where  $\Gamma$  is the functor of global sections. Note that according to this terminology, if  $\phi: Y \rightarrow X$  is a morphism and  $\mathcal{F}$  is a coherent sheaf on  $Y$ , then  $\mathcal{R}\phi_* \mathcal{F}$  is the complex whose cohomology sheaves give rise to the usual higher direct images of  $\mathcal{F}$ .

Similarly, the left derived functor of an additive functor  $F$ , if it exists, is denoted by  $\mathcal{L}F$  and  $\mathcal{L}^i F$  is short for  $h^i \circ \mathcal{L}F$ .

The next two theorems are very important in studying cohomological properties of singular varieties.

**Theorem 7.2** (Grothendieck Duality) [Har66, VII]. *Let  $\phi: Y \rightarrow X$  be a proper morphism between finite dimensional noetherian schemes that admit dualizing complexes. Then for any bounded complex  $G \in D^b(Y)$ ,*

$$\mathcal{R}\phi_* \mathcal{R}\mathcal{H}om_Y(G, \omega_Y^\bullet) \simeq_{\text{qis}} \mathcal{R}\mathcal{H}om_X(\mathcal{R}\phi_* G, \omega_X^\bullet).$$

**Theorem 7.3** (Adjointness of  $\phi_*$  and  $\phi^*$ ) [Har66, II.5.10]. *Let  $\phi : Y \rightarrow X$  be a proper morphism. Then for any bounded complexes  $F \in D^b(X)$  and  $G \in D^b(Y)$ ,*

$$\mathcal{R}\phi_* \mathcal{R}\mathcal{H}om_Y(\mathcal{L}\phi^* F, G) \simeq_{qis} \mathcal{R}\mathcal{H}om_X(F, \mathcal{R}\phi_* G).$$

Vanishing theorems have played a central role in algebraic geometry for the last couple of decades, especially in classification theory. Kollár [Kol87] gives an introduction to the basic use of vanishing theorems as well as a survey of results and applications available at the time. For more recent results one should consult [EV86, EV92, Ein97, Kol97, Smi97, Kov00b, Kov02, Kov03a, Kov03b]. Because of the availability of those surveys, I will only recall statements that are important for the present article. Nonetheless, any discussion of vanishing theorems should start with the fundamental vanishing theorem of Kodaira.

**Theorem 7.4** [Kod53]. *Let  $Y$  be a smooth complex projective variety and  $\mathcal{L}$  an ample line bundle on  $Y$ . Then*

$$H^i(Y, \omega_Y \otimes \mathcal{L}) = 0 \text{ for } i \neq 0.$$

This has been generalized in several ways, but as noted above I will only state what I use in this article. For the many other generalizations the reader is invited to peruse the above references.

The original statement of Kodaira was generalized to allow semi-ample and big line bundles in place of ample ones by Grauert and Riemenschneider.

**Theorem 7.5** [GR70]. *Let  $Y$  be a smooth complex projective variety and  $\mathcal{L}$  a semi-ample and big line bundle on  $Y$ . Then*

$$H^i(Y, \omega_Y \otimes \mathcal{L}) = 0 \text{ for } i \neq 0.$$

This also has a relative version:

**Theorem 7.6** [GR70]. *Let  $Y$  be a smooth complex variety,  $\phi : Y \rightarrow X$  a projective birational morphism, and  $\mathcal{L}$  a semi-ample line bundle on  $Y$ . Then*

$$\mathcal{R}^i \phi_* (\omega_Y \otimes \mathcal{L}) = 0 \text{ for } i \neq 0.$$

By Serre duality both (7.4) and (7.5) has a dual version:

**Theorem 7.7.** *Let  $Y$  be a smooth complex projective variety and  $\mathcal{L}$  a semi-ample and big line bundle on  $Y$ . Then*

$$H^j(Y, \mathcal{L}^{-1}) = 0 \text{ for } j \neq \dim Y.$$

What would be the dual version of (7.6) in the same spirit? Instead of Serre duality one would have to use Grothendieck duality:



Let  $Y$  be a smooth complex variety of dimension  $d$ ,  $\phi : Y \rightarrow X$  a projective morphism, and  $\mathcal{L}$  a semi-ample line bundle on  $Y$ . Then

$$(7.8) \quad \mathcal{R}\mathcal{H}om_X(\mathcal{R}\phi_*(\omega_Y \otimes \mathcal{L}), \omega_X^\bullet) \simeq_{\text{qis}} \simeq_{\text{qis}} \mathcal{R}\phi_* \mathcal{R}\mathcal{H}om_Y(\omega_Y \otimes \mathcal{L}, \omega_Y[d]) \simeq_{\text{qis}} \mathcal{R}\phi_* \mathcal{L}^{-1}[d]$$

In the case of (7.4) and (7.5)  $X = \text{Spec } \mathbb{C}$ , so  $\omega_X^\bullet \simeq_{\text{qis}} \mathbb{C}$ . Then the left hand side is quasi-isomorphic to the dual of  $\mathcal{R}\phi_*(\omega_X \otimes \mathcal{L}) \simeq_{\text{qis}} \phi_*(\omega_X \otimes \mathcal{L}) \simeq H^0(Y, \omega_X \otimes \mathcal{L})$ . Therefore  $h^i(\mathcal{R}\phi_* \mathcal{L}^{-1}[d]) = 0$  for  $i \neq 0$ . This is how (7.7) follows:  $\mathcal{R}^j \phi_* \mathcal{L}^{-1} = H^j(Y, \mathcal{L}^{-1}) = 0$  for  $j \neq d$ .

In the case  $\phi$  is birational there is a shift by  $d$  on both side so the expected dual form of this vanishing would be

$$(7.9) \quad \mathcal{R}^j \phi_* \mathcal{L}^{-1} = 0 \text{ for } j \neq 0.$$

However, this does not always hold. To see this let us consider the simplest semi-ample line bundle,  $\mathcal{O}_Y$ . Then  $\mathcal{R}^i \phi_* \omega_Y = 0$  for  $i \neq 0$  by (7.6), so (7.8) reduces to the following:

$$(7.10) \quad \mathcal{R}\mathcal{H}om_X(\phi_* \omega_Y, \omega_X^\bullet) \simeq_{\text{qis}} \mathcal{R}\phi_* \mathcal{O}_Y[d]$$

Now suppose that  $X$  is normal and  $\omega_Y \simeq \mathcal{O}_Y$ . Then it follows that if (7.9) holds for  $\mathcal{L} = \mathcal{O}_Y$ , then  $\omega_X^\bullet$  has only one non-zero cohomology sheaf and hence  $X$  is Cohen-Macaulay. In other words, if  $X$  is normal, but not CM and  $Y$  has a trivial canonical bundle, then (7.9) does not hold with  $\mathcal{L} = \mathcal{O}_Y$  or more generally with  $\mathcal{L} = \phi^* \mathcal{M}$  for any line bundle  $\mathcal{M}$  on  $X$ .

The point is that the dual form of the relative Grauert-Riemenschneider vanishing theorem is a singularity condition on the target of the morphism in question. Notice that (7.9) follows from (7.10) for  $\mathcal{L} = \mathcal{O}_X$  if  $X$  is Cohen-Macaulay and  $\phi_* \omega_Y \simeq \omega_X$ . It turns out that this defines a very important class of singularities which is the topic of the next section.

## 8. RATIONAL SINGULARITIES

Rational singularities are among the most important classes of singularities. The essence of rational singularities is that their cohomological behavior is very similar to that of smooth points. For instance, vanishing theorems can be easily extended to varieties with rational singularities. Establishing that a certain class of singularities is rational opens the door to using very powerful tools on varieties with those singularities.

**DEFINITION 8.1.** Let  $X$  be a normal variety and  $\phi : Y \rightarrow X$  a resolution of singularities.  $X$  is said to have *rational singularities* if  $\mathcal{R}^i \phi_* \mathcal{O}_Y = 0$  for all  $i > 0$ , or equivalently if the natural map  $\mathcal{O}_X \rightarrow \mathcal{R}\phi_* \mathcal{O}_Y$  is a quasi-isomorphism.

The notion of *irrational centers* is very closely related. For the definition and basic properties see [Kov11c].

A very useful property of rational singularities is that they are Cohen-Macaulay. In fact, this is part of Kempf's characterization of rational singularities:

**Theorem 8.2.** [KKMSD73, p.50] *Let  $X$  be a normal variety and  $\phi : Y \rightarrow X$  a resolution of singularities. Then  $X$  has rational singularities if and only if  $X$  is Cohen-Macaulay and  $\phi_*\omega_Y \simeq \omega_X$ .*

*Proof.* Let  $d = \dim X$ . If  $X$  has rational singularities, then

$$\begin{aligned} \omega_X^\bullet &\simeq_{\text{qis}} \mathcal{R}\mathcal{H}om_X(\mathcal{O}_X, \omega_X^\bullet) \simeq_{\text{qis}} \mathcal{R}\mathcal{H}om_X(\mathcal{R}\phi_*\mathcal{O}_Y, \omega_X^\bullet) \simeq_{\text{qis}} \\ &\simeq_{\text{qis}} \mathcal{R}\phi_*\mathcal{R}\mathcal{H}om_Y(\mathcal{O}_Y, \omega_Y[d]) \simeq_{\text{qis}} \mathcal{R}\phi_*\omega_Y[d] \simeq_{\text{qis}} \phi_*\omega_Y[d], \end{aligned}$$

which implies that  $X$  has to be Cohen-Macaulay and  $\omega_X \simeq \phi_*\omega_Y$ .

Similarly, if  $X$  is Cohen-Macaulay and  $\omega_X \simeq \phi_*\omega_Y$ , then  $\omega_X^\bullet \simeq_{\text{qis}} \phi_*\omega_Y[d]$  and so

$$\begin{aligned} \mathcal{R}\phi_*\mathcal{O}_Y &\simeq_{\text{qis}} \mathcal{R}\phi_*\mathcal{R}\mathcal{H}om_Y(\omega_Y[d], \omega_Y^\bullet) \simeq_{\text{qis}} \mathcal{R}\mathcal{H}om_X(\mathcal{R}\phi_*\omega_Y[d], \omega_X^\bullet) \simeq_{\text{qis}} \\ &\simeq_{\text{qis}} \mathcal{R}\mathcal{H}om_X(\phi_*\omega_Y[d], \omega_X^\bullet) \simeq_{\text{qis}} \mathcal{R}\mathcal{H}om_X(\omega_X^\bullet, \omega_X^\bullet) \simeq_{\text{qis}} \mathcal{O}_X \end{aligned}$$

shows that  $X$  has rational singularities. □

A very important fact is that log terminal singularities are rational:

**Theorem 8.3** [Elk81]. *Let  $X$  be a variety with log terminal singularities. Then  $X$  has rational singularities.*

This is actually an easy consequence of a characterization theorem that will be stated later in (10.2). The proof will be given in (10.6) after the necessary notation is introduced in §10.

In particular, canonical singularities are rational and as a corollary one obtains that the total space of a stable family should have rational singularities.

Now we may repeat the investigation that helped us figure out what kind of singularities stable varieties should have. Previously we figured that if the total space has canonical singularities then the fibers should have semi log canonical singularities. Next we would like to see what it means for the fibers that the total space of the family has rational singularities.

So, let  $f : X \rightarrow B$  be a family of reduced varieties such that  $B$  is a smooth curve and  $X$  has rational singularities. Let  $b \in B$  a fixed point and let  $\phi : Y \rightarrow X$  be a resolution of singularities such that  $\text{supp}(\text{Exc}(\phi) \cup \phi^{-1}X_b)$  is a simple normal crossing divisor. Observe that by assumption and construction  $X_b = f^*b$  is a Cartier divisor and  $Y_b = \phi^*X_b$ . Following the spirit of our assumption on stable families (3.9) assume that  $Y_b$  is reduced, that is,  $Y_b = \phi^{-1}X_b$ . One also has the

following commutative diagram of distinguished triangles:

$$\begin{array}{ccccccc}
 \mathcal{O}_X(-X_b) & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}_{X_b} & \xrightarrow{+1} & \\
 \alpha_1 \downarrow & & \alpha_2 \downarrow & & \alpha_3 \downarrow & & \\
 \mathcal{R}\phi_*\mathcal{O}_Y(-Y_b) & \longrightarrow & \mathcal{R}\phi_*\mathcal{O}_Y & \longrightarrow & \mathcal{R}\phi_*\mathcal{O}_{Y_b} & \xrightarrow{+1} & 
 \end{array}$$

Notice that if the horizontal morphisms in this diagram are the usual natural morphisms, then  $\alpha_3$  is uniquely determined by  $\alpha_1$  and  $\alpha_2$  by (B.2).

As  $X$  has rational singularities,

$$\alpha_2 : \mathcal{O}_X \rightarrow \mathcal{R}\phi_*\mathcal{O}_Y$$

is a quasi-isomorphism. Since  $\mathcal{O}_Y(-Y_b) \simeq \phi^*\mathcal{O}_X(-X_b)$  it follows by the projection formula that

$$\alpha_1 = \alpha_2 \otimes \text{id}_{\mathcal{O}_X(-X_b)} : \mathcal{O}_X(-X_b) \rightarrow \mathcal{R}\phi_*\mathcal{O}_Y(-Y_b) \simeq_{\text{qis}} \mathcal{R}\phi_*\mathcal{O}_Y \otimes \mathcal{O}_X(-X_b)$$

is also a quasi-isomorphism. Therefore the triangulated category version of the 9-lemma (see B) implies that

$$\alpha_3 : \mathcal{O}_{X_b} \rightarrow \mathcal{R}\phi_*\mathcal{O}_{Y_b}$$

is also a quasi-isomorphism. Note that this does not mean that  $X_b$  has rational singularities as  $Y_b$  is not a resolution of singularities, it is in general not even birational to  $X_b$ . However, it definitely means that these singularities are not too far from rational singularities.

We found in §4 that one cannot expect the fibers to have the same type of singularities as the total space, just as one cannot expect all hyperplane sections of varieties in general to have the same type of singularities as the original varieties. Similarly here one cannot expect to have the members of the family have rational singularities. However, just as in §4, one finds that the singularities of the fibers are not too much worse. These are called Du Bois singularities and we will get acquainted with them in the next few sections. Notice that the condition we obtained here is almost identical to the one given by Schwede's criterion in (9.8).

## 9. DB SINGULARITIES

Du Bois singularities are probably harder to appreciate than rational singularities at first, but they are equally important. Their main importance comes from two facts: They are not too far from rational singularities, that is, they share many of their properties, but the class of Du Bois singularities is more inclusive than that of rational singularities. For instance, log canonical singularities are Du Bois, but not necessarily rational.

Du Bois singularities are defined via Deligne's Hodge theory and so their strong connection to the singularities of the minimal model program might seem

unexpected. Nevertheless, they play a very important role. We will need a little preparation before we can define these singularities, but first I would like to mention a few facts to underline their importance.

The concept of Du Bois singularities, abbreviated as *DB*, was introduced by Steenbrink in [Ste83] as a weakening of rationality. The following statement is a direct consequence of the definition and this is the most important property of a DB singularity:

**Theorem 9.1.** *Let  $X$  be a proper scheme of finite type over  $\mathbb{C}$ . If  $X$  has only DB singularities, then the natural map*

$$H^i(X^{\text{an}}, \mathbb{C}) \rightarrow H^i(X^{\text{an}}, \mathcal{O}_{X^{\text{an}}}) \cong H^i(X, \mathcal{O}_X)$$

*is surjective for all  $i$ .*

In fact, this essentially characterizes Du Bois singularities shown by the next theorem. For details see [Kov11b].

**Theorem 9.2.** [Kov11b] *Let  $X$  be a projective variety over  $\mathbb{C}$ . Then  $X$  has only Du Bois singularities if and only if for any  $L \subseteq X$  general (global) complete intersection subvariety*

$$\dim_{\mathbb{C}} H^i(L, \mathcal{O}_L) \leq \dim_{\mathbb{C}} Gr_F^0 H^i(L, \mathbb{C}),$$

*where  $Gr_F^0 H^i(L, \mathbb{C})$  is the graded quotient associated to Deligne's Hodge filtration on  $H^i(L, \mathbb{C})$ .*

Using [DJ74, Lemme 1], (9.1) implies the following:

**Corollary 9.3.** *Let  $f : X \rightarrow B$  be a proper, flat morphism of complex varieties with  $B$  connected. Assume that  $X_b$  has only DB singularities for all  $b \in B$ . Then  $h^i(X_b, \mathcal{O}_{X_b})$  is independent of  $b \in B$  for all  $i$ .*

This will be important later.

The starting point of the precise definition is Du Bois's construction, following Deligne's ideas, of the generalized de Rham complex, which is called the *Deligne-Du Bois complex*. Recall, that if  $X$  is a smooth complex algebraic variety of dimension  $n$ , then the sheaves of differential  $p$ -forms with the usual exterior differentiation give a resolution of the constant sheaf  $\mathbb{C}_X$ . I.e., one has a complex of sheaves,

$$\mathcal{O}_X \xrightarrow{d} \Omega_X^1 \xrightarrow{d} \Omega_X^2 \xrightarrow{d} \Omega_X^3 \xrightarrow{d} \dots \xrightarrow{d} \Omega_X^n \simeq \omega_X,$$

which is quasi-isomorphic to the constant sheaf  $\mathbb{C}_X$  via the natural map  $\mathbb{C}_X \rightarrow \mathcal{O}_X$  given by considering constants as holomorphic functions on  $X$ . Recall that this complex *is not* a complex of quasi-coherent sheaves. The sheaves in the complex are quasi-coherent, but the maps between them are not  $\mathcal{O}_X$ -module morphisms.

Notice however that this is actually not a shortcoming; as  $\mathbb{C}_X$  is not a quasi-coherent sheaf, one cannot expect a resolution of it in the category of quasi-coherent sheaves.

The Deligne-Du Bois complex is a generalization of the de Rham complex to singular varieties. It is a complex of sheaves on  $X$  that is quasi-isomorphic to the constant sheaf,  $\mathbb{C}_X$ . The terms of this complex are harder to describe but its properties, especially cohomological properties are very similar to the de Rham complex of smooth varieties. In fact, for a smooth variety the Deligne-Du Bois complex is quasi-isomorphic to the de Rham complex, so it is indeed a direct generalization.

The construction of this complex,  $\underline{\Omega}_X^\bullet$ , is based on simplicial resolutions. The reader interested in the details is referred to the original article [DB81]. Note also that a simplified construction was later obtained in [Car85] and [GNPP88] via the general theory of polyhedral and cubic resolutions. An easily accessible introduction can be found in [Ste85]. Other useful references are the recent book [PS08] and the survey [KS11b]. I will actually not use these resolutions here. They are needed for the construction, but if one is willing to believe the listed properties (which follow in a rather straightforward way from the construction) then one should be able to follow the material presented here.

Recently Schwede found a simpler alternative construction of (part of) the Deligne-Du Bois complex that does not need a simplicial resolution (9.8). This allows one to define Du Bois singularities (9.5) without needing simplicial resolutions and it is quite useful in applications. For applications of the Deligne-Du Bois complex and Du Bois singularities other than the ones listed here see [Ste83], [Kol95, Chapter 12], [Kov99, Kov00b].

The word “hyperresolution” will refer to either a simplicial, polyhedral, or cubic resolution. Formally, the construction of  $\underline{\Omega}_X^\bullet$  is essentially the same regardless the type of resolution used and no specific aspects of either types will be used.

The following definition is included to make sense of the statements of some of the forthcoming theorems. It can be safely ignored if the reader is not interested in the detailed properties of the Deligne-Du Bois complex and is willing to accept that it is a very close analog of the de Rham complex of smooth varieties.

**Theorem 9.4** [DB81, 6.3, 6.5]. *Let  $X$  be a complex scheme of finite type. Then there exists a unique object  $\underline{\Omega}_X^\bullet \in \text{Ob } D_{\text{filt}}(X)$  such that using the notation*

$$\underline{\Omega}_X^p := Gr_{\text{filt}}^p \underline{\Omega}_X^\bullet[p],$$

*it satisfies the following properties*

(9.4.1)

$$\underline{\Omega}_X^\bullet \simeq_{qis} \mathbb{C}_X.$$

(9.4.2)  $\underline{\Omega}$  is functorial, i.e., if  $\phi: Y \rightarrow X$  is a morphism of complex schemes of finite type, then there exists a natural map  $\phi^*$  of filtered complexes

$$\phi^*: \underline{\Omega}_X^\bullet \rightarrow \mathcal{R}\phi_* \underline{\Omega}_Y^\bullet.$$

Furthermore,  $\underline{\Omega}_X^\bullet \in \text{Ob}\left(D_{\text{filt,coh}}^b(X)\right)$  and if  $\phi$  is proper, then  $\phi^*$  is a morphism in  $D_{\text{filt,coh}}^b(X)$ .

(9.4.3) Let  $U \subseteq X$  be an open subscheme of  $X$ . Then

$$\underline{\Omega}_X^\bullet|_U \simeq_{qis} \underline{\Omega}_U^\bullet.$$

(9.4.4) If  $X$  is proper, then there exists a spectral sequence degenerating at  $E_1$  and abutting to the singular cohomology of  $X$ :

$$E_1^{pq} = \mathbb{H}^q(X, \underline{\Omega}_X^p) \Rightarrow H^{p+q}(X, \mathbb{C}).$$

(9.4.5) If  $\varepsilon_\bullet: X_\bullet \rightarrow X$  is a hyperresolution, then

$$\underline{\Omega}_{X_\bullet}^\bullet \simeq_{qis} R\varepsilon_{\bullet,*} \underline{\Omega}_X^\bullet.$$

In particular,  $h^i(\underline{\Omega}_{X_\bullet}^p) = 0$  for  $i < 0$ .

(9.4.6) There exists a natural map,  $\mathcal{O}_X \rightarrow \underline{\Omega}_X^0$ , compatible with (9.4.2).

(9.4.7) If  $X$  is smooth, then

$$\underline{\Omega}_X^\bullet \simeq_{qis} \Omega_X^\bullet.$$

In particular,

$$\underline{\Omega}_X^p \simeq_{qis} \Omega_X^p.$$

(9.4.8) If  $\phi: Y \rightarrow X$  is a resolution of singularities, then

$$\underline{\Omega}_X^{\dim X} \simeq_{qis} \mathcal{R}\phi_* \omega_Y.$$

It turns out that the Deligne-Du Bois complex behaves very much like the de Rham complex for smooth varieties. Observe that (9.4.4) says that the Hodge-to-de Rham spectral sequence works for singular varieties if one uses the Deligne-Du Bois complex in place of the de Rham complex. This has far reaching consequences and if the associated graded pieces  $\underline{\Omega}_X^p$  turn out to be computable, then this single property leads to many applications.

The natural map  $\mathcal{O}_X \rightarrow \underline{\Omega}_X^0$  given by (9.4.6) may be considered as an invariant of the singularities of  $X$ . Clearly, if  $X$  is smooth, then it is a quasi-isomorphism, but it may be a quasi-isomorphism even if  $X$  is not smooth. In fact, we are interested in situations when this map is a quasi-isomorphism. When  $X$  is proper over  $\mathbb{C}$ , such a quasi-isomorphism implies that the natural map

$$H^i(X^{\text{an}}, \mathbb{C}) \rightarrow H^i(X, \mathcal{O}_X) = \mathbb{H}^i(X, \underline{\Omega}_X^0)$$

is surjective because of the degeneration at  $E_1$  of the spectral sequence in (9.4.4) (cf. (9.1)). Notice that this condition is crucial for proving Kodaira-type vanishing theorems cf. [Kol95, §9], [HK10, 3.H].

Following Du Bois, Steenbrink was the first to study this condition and he christened this property after Du Bois. It should be noted that many of the ideas that play important roles in this theory originated from Deligne. Unfortunately the now standard terminology does not reflect this.

**DEFINITION 9.5.** A scheme  $X$  is said to have *Du Bois singularities* (or *DB singularities* for short) if the natural map  $\mathcal{O}_X \rightarrow \underline{\Omega}_X^0$  from (9.4.6) is a quasi-isomorphism.

**REMARK 9.6.** If  $\varepsilon : X_\bullet \rightarrow X$  is a hyperresolution of  $X$  then  $X$  has Du Bois singularities if and only if the natural map  $\mathcal{O}_X \rightarrow \mathcal{R}\varepsilon_* \mathcal{O}_{X_\bullet}$  is a quasi-isomorphism.

A relative version of this notion for pairs was defined in [Kov11a].

**EXAMPLE 9.7.** It is easy to see that smooth points are Du Bois and Deligne proved that normal crossing singularities are Du Bois as well cf. [DJ74, Lemme 2(b)].

I will finish this section with Schwede's characterization of DB singularities. This condition makes it possible to define DB singularities without hyperresolutions, derived categories, etc. It makes it easier to get acquainted with these singularities, but it is still useful to know the original definition for many applications.

**Theorem 9.8** [Sch07]. *Let  $X$  be a reduced separated scheme of finite type over a field of characteristic zero. Assume that  $X \subseteq Z$  where  $Z$  is smooth and let  $\phi : W \rightarrow Z$  be a proper birational map with  $W$  smooth and where  $Y = \phi^{-1}(X)_{\text{red}}$ , the reduced pre-image of  $X$ , is a simple normal crossings divisor (or in fact any scheme with DB singularities). Then  $X$  has DB singularities if and only if the natural map  $\mathcal{O}_X \rightarrow \mathcal{R}\phi_* \mathcal{O}_Y$  is a quasi-isomorphism.*

*In fact, one can say more. There is a quasi-isomorphism  $\mathcal{R}\phi_* \mathcal{O}_Y \xrightarrow{\simeq_{\text{qis}}} \underline{\Omega}_X^0$  such that the natural map  $\mathcal{O}_X \rightarrow \underline{\Omega}_X^0$  can be identified with the natural map  $\mathcal{O}_X \rightarrow \mathcal{R}\phi_* \mathcal{O}_Y$ .*

Notice that this condition is the one obtained at the end of the previous section. Given that and our earlier findings on (semi-) log canonical singularities it may not come as a surprise that Kollár had conjectured a strong connection between these singularities. As canonical singularities are rational one should expect a similar implication between log canonical and Du Bois:

**Conjecture 9.9** [Kol92, 1.13] (Kollár's Conjecture). *Log canonical singularities are Du Bois.*

This conjecture has been recently confirmed in [KK10]. For more see §10 and in particular (10.15).

## 10. THE SPLITTING PRINCIPLE

The moral of this section can be summarized by the following principle:

**The Splitting Principle.** *Morphisms do not split accidentally.*

REMARK 10.1. It is customary to casually use the word “splitting” to explain the statements of the theorems that follow. However, the reader should be warned that one has to be careful with the meaning of this, because these “splittings” take place in the derived category, which is not abelian. For this reason, in the statements of the theorems below I use the terminology that a morphism admits a *left inverse*. In an abelian category this condition is equivalent to “splitting” and being a direct component (of a direct sum). With a slight abuse of language I labeled these as “Splitting theorems” cf. (10.2), (10.7) and (10.14).

The first theorem I will recall is a criterion for a singularity to be rational.

**Theorem 10.2** [Kov00a] (SPLITTING THEOREM I). *Let  $\phi : Y \rightarrow X$  be a proper morphism of varieties over  $\mathbb{C}$  and  $\varrho : \mathcal{O}_X \rightarrow \mathcal{R}\phi_*\mathcal{O}_Y$  the associated natural morphism. Assume that  $Y$  has rational singularities and  $\varrho$  has a left inverse, i.e., there exists a morphism (in the derived category of  $\mathcal{O}_X$ -modules)  $\varrho' : \mathcal{R}\phi_*\mathcal{O}_Y \rightarrow \mathcal{O}_X$  such that  $\varrho' \circ \varrho$  is a quasi-isomorphism of  $\mathcal{O}_X$  with itself. Then  $X$  has only rational singularities.*

REMARK 10.3. Note that  $\phi$  in the theorem does not have to be birational or even generically finite. It follows from the conditions that it is surjective.

**Corollary 10.4.** *Let  $X$  be a complex variety and  $\phi : Y \rightarrow X$  a resolution of singularities. If  $\mathcal{O}_X \rightarrow \mathcal{R}\phi_*\mathcal{O}_Y$  has a left inverse, then  $X$  has rational singularities.*

**Corollary 10.5.** *Let  $X$  be a complex variety and  $\phi : Y \rightarrow X$  a finite morphism. If  $Y$  has rational singularities, then so does  $X$ .*

Using this criterion it is quite easy to prove that log terminal singularities are rational (8.3). For related statements see [KM98, 5.22] and the references therein.

**Theorem 10.6** (= Theorem 8.3). *Let  $X$  be a variety with log terminal singularities. Then  $X$  has rational singularities.*

*Proof.* [Kov00a] The question is local, so one may restrict to a neighbourhood of a point. Then the index 1 cover  $\pi : \tilde{X} \rightarrow X$  is a finite morphism onto  $X$ . In particular,  $\pi_*$  is exact and the natural morphism  $\mathcal{O}_X \rightarrow \pi_*\mathcal{O}_{\tilde{X}}$  has a left inverse by the construction of the index 1 cover.

Therefore, by (10.5) it is enough to prove that  $\tilde{X}$  has rational singularities and so one may assume that  $X$  has canonical singularities and  $\omega_X$  is a line bundle.

Let  $\phi : Y \rightarrow X$  be a resolution of singularities of  $X$ . Since  $X$  has canonical singularities and  $\omega_X$  is a line bundle, there exists a non-trivial morphism

$$\iota : \mathcal{L}\phi^*\omega_X \simeq_{\text{qis}} \phi^*\omega_X \rightarrow \omega_Y.$$

Its adjoint morphism on  $X$ ,  $\omega_X \rightarrow \mathcal{R}\phi_*\omega_Y$ , is a quasi-isomorphism by (3.5) and (7.6)



Applying  $\mathcal{R}\mathcal{H}om_Y(\_, \omega_Y)$  to  $\iota$  and using (7.3), one obtains the following diagram which defines  $\varrho'$ :

$$\begin{array}{ccccc}
 \mathcal{R}\phi_* \mathcal{R}\mathcal{H}om_Y(\omega_Y, \omega_Y) & \xrightarrow{\simeq_{\text{qis}}} & \mathcal{R}\phi_* \mathcal{R}\mathcal{H}om_Y(\mathcal{L}\phi^* \omega_X, \omega_Y) & \xrightarrow{\simeq_{\text{qis}}} & \mathcal{R}\mathcal{H}om_X(\omega_X, \mathcal{R}\phi_* \omega_Y) \\
 \uparrow \simeq_{\text{qis}} & & & & \downarrow \simeq_{\text{qis}} \\
 \mathcal{R}\phi_* \mathcal{O}_Y & \xrightarrow{\varrho'} & & & \mathcal{O}_X.
 \end{array}$$

The last quasi-isomorphism uses the fact that  $\mathcal{R}\phi_* \omega_Y \simeq_{\text{qis}} \omega_X$ . It is easy to see that  $\varrho' \circ \varrho$  acts trivially on  $\mathcal{O}_X$  and hence the statement follows by 10.2 (or (10.4)).  $\square$

There is a criterion for DB singularities that is similar to the one in (10.2):

**Theorem 10.7** [Kov99, 2.3] (SPLITTING THEOREM II). *Let  $X$  be a complex variety. If  $\mathcal{O}_X \rightarrow \underline{\Omega}_X^0$  has a left inverse, then  $X$  has DB singularities.*

This criterion has several important consequences. Here is one of them:

**Corollary 10.8** [Kov99, 2.6]. *Let  $X$  be a complex variety with rational singularities. Then  $X$  has DB singularities.*

*Proof.* Let  $\phi : Y \rightarrow X$  be a resolution of singularities. Then since  $Y$  is smooth the natural map  $\varrho : \mathcal{O}_X \rightarrow \mathcal{R}\phi_* \mathcal{O}_Y$  factors through  $\underline{\Omega}_X^0$  by (9.4.6). Then, since  $X$  has rational singularities,  $\varrho$  is a quasi-isomorphism, so one obtains that the natural map  $\mathcal{O}_X \rightarrow \underline{\Omega}_X^0$  has a left inverse. Therefore,  $X$  has DB singularities by (10.7).  $\square$

Recently a few more criterions have been found for DB singularities. The next one resembles Kempf's criterion for rational singularities (8.2) and shows that indeed DB singularities may be considered a close generalization of rational singularities.

**Theorem 10.9** [KSS10, 3.1]. *Let  $X$  be a normal Cohen-Macaulay scheme of finite type over  $\mathbb{C}$ . Let  $\phi : Y \rightarrow X$  be a resolution of singularities such that the (reduced) exceptional set  $G$  is a simple normal crossing divisor. Then  $X$  has DB singularities if and only if  $\phi_* \omega_Y(G) \simeq \omega_X$ .*

Related results have been obtained in the non-normal Cohen-Macaulay case, see [KSS10] for details.

**REMARK 10.10.** The submodule  $\phi_* \omega_Y(G) \subseteq \omega_X$  is independent of the choice of the log resolution. Thus this submodule may be viewed as an invariant that partially measures how far a scheme is from being DB (compare with [Fuj08]).

As an easy corollary, one obtains another proof that rational singularities are DB (this time via the Kempf-criterion for rational singularities).

**Corollary 10.11** [Kov00a]. *Let  $X$  be a complex variety with rational singularities. Then  $X$  has DB singularities.*

*Proof.* Since  $X$  has rational singularities, it is Cohen-Macaulay and normal. Then  $\phi_*\omega_Y = \omega_X$  but one also has  $\phi_*\omega_Y \subseteq \phi_*\omega_Y(G) \subseteq \omega_X$ , and thus  $\phi_*\omega_Y(G) = \omega_X$  as well. The statement now follows from Theorem 10.9.  $\square$

One also sees immediately that log canonical singularities coincide with DB singularities in the Gorenstein case.

**Corollary 10.12** [Kov99, 3.6][KSS10, 3.16]. *Suppose that  $X$  is Gorenstein and normal. Then  $X$  is DB if and only if  $X$  is log canonical.*

*Proof.*  $X$  is easily seen to be log canonical if and only if  $\phi_*\omega_{Y/X}(G) \simeq \mathcal{O}_X$ . The projection formula then completes the proof.  $\square$

In fact, a slightly jazzed up version of this argument can be used to show that every Cohen-Macaulay log canonical pair is DB:

**Corollary 10.13** [KSS10, 3.16]. *CM log canonical singularities are DB.*

We will see below that it is actually not necessary to assume CM in the previous theorem. However, the characterization of DB singularities in (10.9) is still useful on its own.

**Theorem 10.14** [KK10, 1.6] (SPLITTING THEOREM III). *Let  $\phi : Y \rightarrow X$  be a proper morphism between reduced schemes of finite type over  $\mathbb{C}$ . Let  $W \subseteq X$  be a closed reduced subscheme with ideal sheaf  $\mathcal{I}_{W \subseteq X}$  and  $F := \phi^{-1}(W) \subset Y$  with ideal sheaf  $\mathcal{I}_{F \subseteq Y}$ . Assume that the natural map  $\varrho$*

$$\begin{array}{ccc} & \varrho' & \\ \swarrow & \xrightarrow{\quad} & \searrow \\ \mathcal{I}_{W \subseteq X} & \xrightarrow[\varrho]{} & \mathcal{R}\phi_*\mathcal{I}_{F \subseteq Y} \end{array}$$

*admits a left inverse  $\varrho'$ , that is,  $\varrho' \circ \varrho = \text{id}_{\mathcal{I}_{W \subseteq X}}$ . Then if  $Y, F$ , and  $W$  all have DB singularities, then so does  $X$ .*

A somewhat more general version of this was proved in [Kov11d].

This criterion forms the cornerstone of the proof of the following theorem:

**Theorem 10.15** [KK10, 1.5]. *Let  $\phi : Y \rightarrow X$  be a proper surjective morphism with connected fibers between normal varieties. Assume that  $Y$  has log canonical singularities and  $K_Y \sim_{\mathbb{Q}, \phi} 0$ . Then  $X$  is DB.*

**Corollary 10.16** [KK10, 1.4]. *Log canonical singularities are DB.*

For the proofs and more general statements, please see [KK10]. Also, note that this statement holds in a more general situation, namely it is in fact true that already semi-log canonical singularities are DB. This is proved in [Kol11].

REMARK 10.17. Notice that in (10.14) it is not required that  $\phi$  be birational. On the other hand the assumptions of the theorem and [Kov00a, Thm 1] imply that if  $Y \setminus F$  has rational singularities, e.g., if  $Y$  is smooth, then  $X \setminus W$  has rational singularities as well.

This theorem is used in [KK10] to derive various consequences, some of which are formally unrelated to DB singularities. I will mention some of these in the sequel, but the interested reader should look at the original article to obtain the full picture.

## 11. STABLE FAMILIES

The connection between log canonical and DB singularities has many useful applications in moduli theory. I list a few below without proof.

**Theorem 11.1** [KK10, 7.8,7.9,7.13]. *Let  $f : X \rightarrow B$  be a flat projective morphism of complex varieties with  $B$  connected and such that  $X_b$  has log canonical singularities for all  $b \in B$ . Then*

- (11.1.1)  $h^i(X_b, \mathcal{O}_{X_b})$  is independent of  $b \in B$  for all  $i$ .
- (11.1.2) If one fiber of  $f$  is Cohen-Macaulay, then all fibers are Cohen-Macaulay.
- (11.1.3) The cohomology sheaves  $h^i(\omega_f^\bullet)$  are flat over  $B$ , where  $\omega_f^\bullet$  denotes the relative dualizing complex of  $f$ .

For arbitrary flat, proper morphisms, the set of fibers that are Cohen-Macaulay is open, but not necessarily closed. Thus the key point of (11.1.2) is to show that this set is also closed.

The generalization of these results to the semi log canonical case turns out to be straightforward, but it needs some foundational work to extend some of the results used here to the semi log canonical case. This is done in [Kol]. The general case then implies that each connected component of the moduli space of stable log varieties parameterizes either only Cohen-Macaulay or only non-Cohen-Macaulay objects.

Notice that this still does not mean that one should abandon the non-Cohen-Macaulay objects. There exists smooth projective varieties of general type whose log canonical model is not Cohen-Macaulay and one should naturally prefer to have a moduli space that includes these. Nevertheless, it is very useful to know that if the general fiber is Cohen-Macaulay, then so is the special fiber.

(11.1) is proved using (10.15), (9.3) and the following theorem. Before I can state that theorem I need a simple definition. Let  $f : X \rightarrow B$  be a flat morphism. One says that  $f$  is a *DB family* if  $X_b$  is DB for all  $b \in B$ .

**Theorem 11.2** [KK10, 7.9]. *Let  $f : X \rightarrow B$  be a projective DB family and  $\mathcal{L}$  a relatively ample line bundle on  $X$ . Then*

- (11.2.1) the sheaves  $h^{-i}(\omega_f^\bullet)$  are flat over  $B$  for all  $i$ ,
- (11.2.2) the sheaves  $f_*(h^{-i}(\omega_f^\bullet) \otimes \mathcal{L}^{\otimes q})$  are locally free and compatible with arbitrary base change for all  $i$  and for all  $q \gg 0$ , and
- (11.2.3) for any base change morphism  $\vartheta : T \rightarrow B$  and for all  $i$ ,

$$(h^{-i}(\omega_f^\bullet))_T \simeq h^{-i}(\omega_{f_T}^\bullet).$$

Let me emphasize a special case of this theorem. This had been known for families of CM varieties, so for instance for stable families of relative dimension at most 2.

**Corollary 11.3.** *Let  $f : X \rightarrow B$  be a weakly stable family. Then  $\omega_{X/B}$  commutes with arbitrary base change.*

For related results that show that this statement is sharp in a certain sense see [Pat10].

## 12. DEFORMATIONS OF DB SINGULARITIES

Given the importance of DB singularities in moduli theory it is a natural question whether they are invariant under small deformation.

It is relatively easy to see from the construction of the Deligne-Du Bois complex that a general hyperplane section (or more generally, the general member of a base point free linear system) on a variety with DB singularities again has DB singularities. Therefore the question of deformation follows from the following.

**Conjecture 12.1** [Ste83]. *Let  $D \subset X$  be a reduced Cartier divisor and assume that  $D$  has only DB singularities in a neighborhood of a point  $x \in D$ . Then  $X$  has only DB singularities in a neighborhood of the point  $x$ .*

This conjecture was confirmed for isolated Gorenstein singularities by Ishii [Ish86]. Also note that rational singularities satisfy this property, see [Elk78].

One also has the following easy corollary of the results presented earlier:

**Theorem 12.2.** *Assume that  $X$  is Gorenstein and  $D$  is normal. Then the statement of (12.1) is true.*

*Proof.* The question is local so one may restrict to a neighborhood of  $x$ . If  $X$  is Gorenstein, then so is  $D$  as it is a Cartier divisor. Then  $D$  is log canonical by (10.12), and then  $X$  is also log canonical by inversion of adjunction [Kaw07]. (Recall that if  $D$  is normal, then so is  $X$  along  $D$ ). Therefore  $X$  is also DB.  $\square$

REMARK 12.3. It is claimed in [Kov00a, 3.2] that the conjecture holds in full generality. Unfortunately, the proof published there is not complete. It works as long as one assumes that the non-DB locus of  $X$  is contained in  $D$ . For instance, one may assume that this is the case if the non-DB locus is isolated.

The problem with the proof is the following: it is stated that by taking hyperplane sections one may assume that the non-DB locus is isolated. However, this is incorrect. One may only assume that the *intersection* of the non-DB locus of  $X$  with  $D$  is isolated. If one takes a further general section then it will miss the intersection point and then it is not possible to make any conclusions about that case.

Until very recently the best known result with regard to this conjecture had been the following:

**Theorem 12.4** [Kov00a, 3.2]. *Let  $D \subset X$  be a reduced Cartier divisor and assume that  $D$  has DB singularities in a neighborhood of a point  $x \in D$  and that  $X \setminus D$  has DB singularities. Then  $X$  has only DB singularities in a neighborhood of  $x$ .*

After submitting this article, but fortunately before it went to press the above conjecture has been settled by the author of this paper and Karl Schwede:

**Theorem 12.5** [KS11a, 4.1,4.2]. *Conjecture 12.1 holds, that is: If  $D \subset X$  is a reduced Cartier divisor such that  $D$  has only DB singularities in a neighborhood of a point  $x \in D$ , then  $X$  has only DB singularities in a neighborhood of the point  $x$ .*

Experience shows that divisors not in general position tend to have worse singularities than the ambient space in which they reside. Therefore one would in fact expect that if  $X \setminus D$  and  $D$  are nice (e.g., they have DB singularities), then perhaps  $X$  is even better behaved.

We have also seen that rational singularities are DB and at least Cohen-Macaulay DB singularities are not so far from being rational cf. (10.9). The following result of Schwede supports this philosophical point.

**Theorem 12.6** [Sch07, 5.1]. *Let  $X$  be a reduced scheme of finite type over a field of characteristic zero,  $D$  a Cartier divisor that has DB singularities and assume that  $X \setminus D$  is smooth. Then  $X$  has rational singularities (in particular, it is Cohen-Macaulay).*

Let me conclude with a conjectural generalization of this statement:

**Conjecture 12.7.** *Let  $X$  be a reduced scheme of finite type over a field of characteristic zero,  $D$  a Cartier divisor that has DB singularities and assume that  $X \setminus D$  has rational singularities. Then  $X$  has rational singularities (in particular, it is Cohen-Macaulay).*

Essentially the same proof as in (12.2) shows that this is also true under the same additional hypotheses.

**Theorem 12.8.** *Assume that  $X$  is Gorenstein and  $D$  is normal. Then the statement of (12.7) is true.*

*Proof.* If  $X$  is Gorenstein, then so is  $D$  as it is a Cartier divisor. Then by (10.12)  $D$  is log canonical. Then  $X$  is also log canonical near  $D$  by inversion of adjunction [Kaw07].

As  $X$  is Gorenstein and  $X \setminus D$  has rational singularities, it follows that  $X \setminus D$  has canonical singularities. Then  $X$  has only canonical singularities everywhere. This can be seen by observing that  $D$  is a Cartier divisor and examining the discrepancies that lie over  $D$  for  $(X, D)$  as well as for  $X$ . Therefore, by (8.3) [Elk81]  $X$  has only rational singularities along  $D$ .  $\square$

### APPENDIX A. THE $\mathbb{Q}$ -CARTIER CONDITION IN FAMILIES

Let  $R \subseteq \mathbb{P}^4$  be a quartic rational normal curve, i.e., the image of the embedding of  $\mathbb{P}^1$  into  $\mathbb{P}^4$  by the global sections of  $\mathcal{O}_{\mathbb{P}^1}(4)$ .

Let  $T \subseteq \mathbb{P}^5$  be a quartic rational scroll, i.e., the image of the embedding of  $\mathbb{P}^1 \times \mathbb{P}^1$  into  $\mathbb{P}^5$  by the global sections of  $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 2)$ . Then  $R$  is a hyperplane section of  $T$ . Indeed, let  $f_1$  and  $f_2$  denote the divisor classes of the two rulings on  $T$  and let  $H \subseteq \mathbb{P}^5$  be a general hyperplane. Then  $C := H \cap T$  is a smooth curve such that  $C \sim_T f_1 + 2f_2$ . Then by the adjunction formula  $2g(C) - 2 = (-2f_1 - 2f_2 + C) \cdot C = -2$ , hence  $C \simeq \mathbb{P}^1$ . Furthermore, then  $C^2 = 4$ , so  $\mathcal{O}_T(1, 2)|_C \simeq \mathcal{O}_C(4)$ . Therefore  $C$  is a quartic rational curve in  $H \simeq \mathbb{P}^4$ , and thus it may be identified with  $R$ .

Let  $C_R \subseteq \mathbb{P}^5$  be the projectivized cone over  $R$  in  $\mathbb{P}^5$  and  $C_T \subseteq \mathbb{P}^6$  the projectivized cone over  $T$  in  $\mathbb{P}^6$ . Then as  $R$  is a hyperplane section of  $T$ , it follows that both  $T$  and  $C_R$  are hyperplane sections of  $C_T$ , so  $T$  is a smoothing of  $C_R$ .

Let  $V \subseteq \mathbb{P}^5$  be a Veronese surface, i.e., the image of the Veronese embedding; the embedding of  $\mathbb{P}^2$  into  $\mathbb{P}^5$  by the global sections of  $\mathcal{O}_{\mathbb{P}^2}(2)$ . Let  $D \subset V$  be the image of a smooth conic of  $\mathbb{P}^2$ . Then  $D$  is a hyperplane section of  $V$  and it is also a rational normal quartic curve in  $\mathbb{P}^4$  so it can also be identified with  $R$ . Therefore, the same way as above, using  $C_V$ , the cone over  $V$ , one sees that  $V$  is also a smoothing of  $C_R$ .

It is relatively easy, and thus left to the reader, to compute that  $C_R$  has log terminal singularities. In particular, this type of singularity is among those that appear on stable varieties. In fact, considering a cyclic covering [KM98, 2.50] branched over a highly divisible relatively very ample divisor gives a family of stable varieties with the same kind of singularities as the ones that appear here. This can be applied for both of the families coming from  $C_T$  and  $C_V$ .

The problem this example points to is that if one allows arbitrary families, then one may get unwanted results. For example, using the families derived from  $C_T$  and  $C_V$  would mean that  $T \simeq \mathbb{P}^1 \times \mathbb{P}^1$  and  $V \simeq \mathbb{P}^2$  should be considered to have the same deformation type (or the same statement for the surfaces of general type on the cyclic cover mapping to these fibers). However, there are obviously no smooth families that they both belong to, they are topologically very different. For instance,  $K_T^2 = 8$  while  $K_V^2 = 9$ .

The crux of the matter is that  $K_{C_T}$  is *not*  $\mathbb{Q}$ -Cartier and consequently the family obtained from it is not a (weakly) stable family as defined in (6.20) and (6.22). This is actually an important point: the canonical classes of the members of the family are  $\mathbb{Q}$ -Cartier, but the relative canonical class of the family is not  $\mathbb{Q}$ -Cartier. In particular, the canonical divisors of the members of the family are not consistent.

The family obtained from  $C_V$  has a  $\mathbb{Q}$ -Cartier canonical class and consequently ensures that the canonical divisors of the members of the family are similar

to some extent. Among other things this implies that  $K_{C_R}^2 = 9$ . One may also use an actual parametrization of  $C_R$  to verify this fact independently. It is interesting to note that  $K_{C_R}$  is  $\mathbb{Q}$ -Cartier, but not Cartier even though its self-intersection number is an integer.

#### APPENDIX B. THE NINE LEMMA IN TRIANGULATED CATEGORIES

For lack of an appropriate reference the following pseudo-trivial theorem is proved here for the reader's convenience.

**Theorem B.1.** *Let  $A, B, C, A', B', C', A'', B''$  be objects in a triangulated category  $\mathfrak{T}$  and assume that there exists a commutative diagram in which the first two rows and the first two columns form distinguished triangles:*

$$(B.1.1) \quad \begin{array}{ccccc} A & \xrightarrow{\phi} & B & \xrightarrow{\psi} & C \xrightarrow{+1} \rightarrow \\ \alpha \downarrow & & \beta \downarrow & & \\ A' & \xrightarrow{\phi'} & B' & \xrightarrow{\psi'} & C' \xrightarrow{+1} \rightarrow \\ \alpha' \downarrow & & \beta' \downarrow & & \\ A'' & & B'' & & \\ +1 \downarrow & & +1 \downarrow & & \\ & & & & \end{array}$$

*Then there exist a morphism  $\gamma : C \rightarrow C'$  with mapping cone  $C''$ , i.e., such that  $C \longrightarrow C' \longrightarrow C'' \xrightarrow{+1} \rightarrow$  is a distinguished triangle, and morphisms  $A'' \rightarrow B''$ ,  $B'' \rightarrow C''$ ,  $C'' \rightarrow A''[1]$  such that*

$$A'' \longrightarrow B'' \longrightarrow C'' \xrightarrow{+1} \rightarrow$$

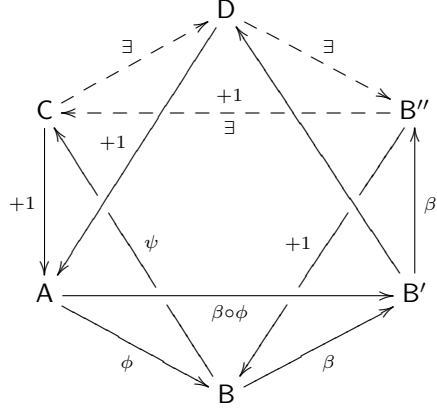
*is a distinguished triangle, and the diagram*

$$(B.1.2) \quad \begin{array}{ccccc} A & \xrightarrow{\phi} & B & \xrightarrow{\psi} & C \xrightarrow{+1} \rightarrow \\ \alpha \downarrow & & \beta \downarrow & & \gamma \downarrow \\ A' & \xrightarrow{\phi'} & B' & \xrightarrow{\psi'} & C' \xrightarrow{+1} \rightarrow \\ \alpha' \downarrow & & \beta' \downarrow & & \gamma' \downarrow \\ A'' & \longrightarrow & B'' & \longrightarrow & C'' \xrightarrow{+1} \rightarrow \\ +1 \downarrow & & +1 \downarrow & & +1 \downarrow \\ & & & & \end{array}$$

is commutative. Furthermore, if the triangulated category  $\mathfrak{T}$  is a derived category and  $C$  and  $C'$  are such that  $h^i(C) = 0$  for  $i \neq 0$  and  $h^j(C') = 0$  for  $j < 0$ , then  $\gamma$  is uniquely determined by the original diagram (B.1.1).

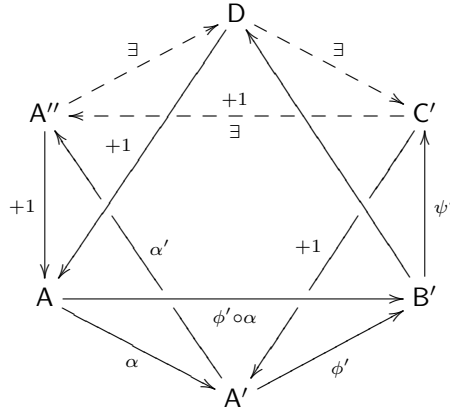
*Proof.* The proof consists of repeated applications of the octahedral axiom.

First consider the composition  $A \rightarrow B \rightarrow B'$  and let  $D$  be an object that completes this morphism to a distinguished triangle.



Then by the octahedral axiom there exist morphisms as indicated on the above diagram such that  $C \longrightarrow D \longrightarrow B \xrightarrow{+1}$  is a distinguished triangle.

Next, consider the composition  $A \rightarrow A' \rightarrow B'$ . Since  $\phi' \circ \alpha = \beta \circ \phi$ ,  $D$  is still an object that completes this composition to a distinguished triangle.

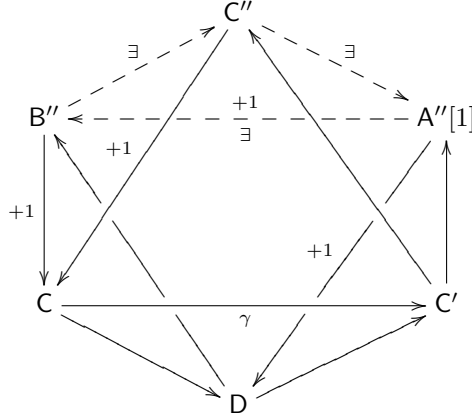


Then by the octahedral axiom there exist morphisms as indicated on the above diagram such that  $A'' \longrightarrow D \longrightarrow C' \xrightarrow{+1}$  is a distinguished triangle.

Finally, consider the composition  $C \rightarrow D \rightarrow C'$  using the morphisms obtained by the above two applications of the octahedral axiom. Let  $\gamma : C \rightarrow C'$  be defined



as this composition and  $C''$  its mapping cone.



Then by the octahedral axiom there exist morphisms as indicated on the above diagram such that  $B'' \longrightarrow C'' \longrightarrow A''[1] \xrightarrow{+1}$ , and hence

$$A'' \longrightarrow B'' \longrightarrow C'' \xrightarrow{+1}$$

are distinguished triangles. The fact that the diagram (B.1.2) is commutative follows from the construction and the uniqueness of  $\gamma$  in the indicated case follows from Lemma B.2.  $\square$

**Lemma B.2.** [KK10, 2.2.4] *Let  $C, C'$  objects in a derived category such that  $h^i(C) = 0$  for  $i \neq 0$  and  $h^j(C') = 0$  for  $j < 0$ . Then any morphism  $\gamma : C \rightarrow C'$  is uniquely determined by  $h^0(\gamma)$ .*

*Proof.* By the assumption, the morphism  $\gamma : C \rightarrow C'$  may be represented by a morphism of complexes  $\tilde{\gamma} : \tilde{C} \rightarrow \hat{C}$ , where  $C \simeq \tilde{C}$  such that  $\tilde{C}^0 = h^0(C)$  and  $\tilde{C}^i = 0$  for all  $i \neq 0$ , and  $C' \simeq \hat{C}$  such that  $h^0(\hat{C}) \subseteq \hat{C}^0$ . However  $\tilde{\gamma}$  has only one non-zero term,  $h^0(\gamma)$ .  $\square$

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